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Gaussian n-Markovian Processes and Stochastic Boundary Value Problems

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Introduction

In this paper a class of Gaussian processes on the interval is considered. The processes arise naturally as formal solutions of generalized stochastic boundary value problems with Gaussian white noise input. This is briefly explained in preliminary Sect. 1. Such processes were also studied in connection with interpolation by generalized spline functions [8, 9].

We give the characterization of those boundary value problems for which the solutions are n-Markovian (Sect. 2). The proof is based on the characterization of Markov property in terms of reproducing kernel Hilbert space associated to a Gaussian process similar to the one given in e.g. [7] or [5].

1. Stochastic Boundary Value Problems

Let $H^n(I)$, $n \geqq 1$ denote the Sobolev space of real valued functions defined on the interval $I=[0, 1]$ such that $f \in C^{n-1}(I)$, $D^{n-1}f$ being absolutely continuous with $D^n f \in L^2(I)$. Next let L denote the following ordinary differential operator

$$Lf(t)=D^n f(t)+a_{n-1}(t)D^{n-1}f(t)+\ldots+a_1(t)Df(t)+a_0(t), \qquad D=\frac{d}{dt}$$

where $a_i \in C^\infty(I)$ and let $\Lambda=(\lambda_i,\ 1 \leqq i \leqq n)$ be the set of linear functionals on $H^n(I)$ of the form

$$\lambda(f)=\sum_{i=0}^{n-1} \int_I D^i f(t)\, dv_i(t)$$

where v_i are some functions of bounded variation. In what follows Λ will be treated either as a sequence of functionals or the linear operator on $H^n(I)$ with values in R^n. It is assumed that λ_i, $1 \leqq i \leqq n$, are lineary independent on $\ker L$.

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Let $w=(w(t),\ t\in I)$ be the standard Brownian motion on some probability space (Ω, F, P). By W we denote the distributional derivative of w treated as a generalized process. It is well known that $W=(W(\varphi),\ \varphi\in C_0^\infty(I))$ is a generalized process with the covariance functional

$$E(W(\varphi)\,W(\psi))=\int \varphi(t)\,\psi(t)\,dt.$$

Suppose we have the sequence $\xi=(\xi_1,\ldots,\xi_n)$ of random variables on (Ω, F, P) and consider the following boundary value problem

$$LX=W,\quad \Lambda X=\xi. \tag{1}$$

Let now $X=(X(t),\ t\in I)$ be the process defined by formula

$$X(t)=\int_0^1 G(t,s)\,dw(s)+\sum_{i=1}^n \xi_i z_i(t) \tag{2}$$

where $G(t,s)$, $t,s\in I$, is the Green function for the boundary value problem $Lf=g$, $\Lambda f=0$, and $(z_i,\ 1\leqq i\leqq n)$ is a basis in $\ker L$ dual to (λ_j). For each $t\in I$, $\int G(t,s)\,dw(s)$ is a stochastic integral of Paley Wiener Zygmund type. We assume that the versions of the integrals are chosen in such a way that X is separable. The process X is easily seen to be stochasticely continuous hence by Proposition III.4.2 [6] it is separable with respect to any countable dense subset of I. Each such version will be called the (formal) solution of (1). This is motivated by the fact that there exist separable versions of X given by (2) satisfying (1) in a prescribed sense e.g. distributionally almost surely [3].

Proposition 1. *If X is the solution of* (1) *then*

$$P\{X\in C^{n-1}(I)\}=1.$$

Proof. The Green function in (2) is given by (see [4])

$$G(t,s)=g(t,s)-\sum_{i=1}^n z_i(t)\,\lambda_i g(\cdot,s)$$

where g is the Green function for the initial value problem $Lf=F$, $D^i f(0)=0$ for $0\leqq i\leqq n-1$. Since $z_i\in C^n(I)$ for $1\leqq i\leqq n$, it is enough to prove that the process $y(t)=\int g(t,s)\,dw(s)$ is $n-1$ times continuously differentiable. If $n=1$ it is easy to check that $E|y(t+h)-y(t)|^2\leqq \text{const}\,h$ and Kolmogorov's theorem can be applied. For $n\geqq 2$ at first one checks that

$$\int_I |\Delta_h^2 f(t,s)|^2\,ds\leqq \text{const}\cdot h^3$$

where $f(t,s)=D^{(n-2,0)}g(t,s)$ and Δ_h^2 denotes the progressive difference of order two applied to the first variable. Having this it is easy to verify that

$$E|\Delta_h^n y(t)|^2=\int_I |\Delta_h^n g(t,s)|^2\,ds\leqq \text{const}\cdot h^{2n-1}$$

with some constant independent of t. Now Theorem 4.2 of [2] can be applied to obtain $P\{X \in C^{n-1}(I)\} = 1$.

In what follows it is assumed that in (2) $\xi = (\xi_1, \ldots, \xi_n)$ is a set of independent Gaussian random variables (possibly degenerated) with mean zero and such that W is independent of ξ. For notational convenience it is assumed as well that the random variables ξ_k are numbered in such a way that

$$b_k = b_k(\xi) = E(\xi_k^2) \begin{cases} > 0 & \text{for } 1 \leqq k \leqq m \\ = 0 & \text{for } m < k \leqq n. \end{cases} \tag{3}$$

Moreover we denote

$$\Lambda' = (\lambda_1, \ldots, \lambda_m) \qquad \Lambda'' = (\lambda_{m+1}, \ldots, \lambda_n).$$

It is convenient to introduce the following relation. We say that the set of functionals Λ and μ are equivalent with respect to L and $(b_1, \ldots, b_n)$, $(\Lambda \sim \mu)$, iff the processes defined according to (2) have the same covariance function.

2. Markov Property

Let us recall that the process X is called n-Markovian iff $P\{X \in C^{n-1}(I)\} = 1$ and the n-dimensional process given by $\overline{X}(t) = (X(t), DX(t), \ldots, D^{n-1}X(t))$ is Markovian in the usual sense.

Theorem 1. *The Gaussian process X defined by (2) is n-Markovian if and only if the following condition (S) holds: there exist sequences of points $t_1, \ldots, t_n \in I$ and linear functionals $\mu = (\mu_1, \ldots, \mu_n)$ on $H^n(I)$ such that $\Lambda \sim \mu$ and*

$$\operatorname{supp} \mu_i = \{t_i\} \qquad \text{for } 1 \leqq i \leqq n.$$

At first we reduce the theorem with the help of Proposition 2 and 3 below.

For a Gaussian process X let us denote by $H(X)$ the reproducing kernel Hilbert space ([1]) generated by the covariance function $R(t, s) = E(X(t)X(s))$ of X. It is the real Hilbert space spanned by $R(t, \cdot)$, $t \in I$, with respect to the scalar product $(\ ,\ )$ determined by

$$(R(t, \cdot), R(s, \cdot)) = R(t, s).$$

The proof of the theorem is based on the following characterization of the Markov property.

Proposition 2. *A Gaussian process $X = (X(t),\ t \in I)$ with mean value zero is n-Markovian if and only if $P\{X \in C^{n-1}(I)\} = 1$ and the following pair of conditions for $H(X)$ holds*

$$\forall\, t \in I \quad \text{if } f \in H(X) \text{ and } D^i f(t) = 0 \text{ for } 0 \leqq i \leqq n-1 \text{ then } f \cdot 1_{[0,t]} \text{ and } f \cdot 1_{[t,1]} \in H(X), \tag{4}$$

$$\forall\, t \in I \quad \text{if } f, g \in H(X) \text{ with } \operatorname{supp} f \subset [0, t],\ \operatorname{supp} g \subset [t, 1] \text{ then } (f, g) = 0. \tag{5}$$

The proof is omitted since it can be obtained by a suitable modification of the proof of Theorem 3.3 [7].

If R is the covariance function of the process given by (2) then since ξ and W are independent we have

$$R(t,s)=\sum_{i=1}^{m} b_i z_i(t) z_i(s)+\int G(t,u)\,G(s,u)\,du$$

where (b_i) and m are given as in (3).

From Theorem 3.1 of [4] it is easy to deduce the following

Proposition 3. *The function $R(t,s)$ $t,s\in I$ is the reproducing kernel for the Hilbert space*

$$H_\xi^A=\{f\in H^n(I);\ \lambda_i(f)=0 \text{ for } m<i\leqq n\}=\ker \Lambda''$$

with the scalar product

$$(f,g)_L=\sum_{i=1}^{m} b_i^{-1}\lambda_i(f)\lambda_i(g)+\int (Lf)(Lg)$$

and m defined as in (3).

According to Propositions 2 and 3 it is enough to show that the condition (S) of the theorem is equivalent to the pair of conditions (4) and (5) for the Hilbert space H_ξ^A but this follows easily from the following

Lemma. *Condition (S) of Theorem 1 is equivalent to the following set of conditions*

$$\forall\, t\in I \quad \text{if } f\in\ker\Lambda'' \text{ and } D^i f(t)=0 \text{ for } 0\leqq i\leqq n-1 \text{ then } f\cdot 1_{[0,t]} \text{ and } f\cdot 1_{[t,1]} \text{ are both in } \ker\Lambda'', \tag{6}$$

$$\forall\, t\in I \quad \text{if } f,g\in\ker\Lambda'' \text{ and } \operatorname{supp} f\subset[0,t],\ \operatorname{supp} g\subset[t,1] \text{ then } \sum_{i=1}^{m} b_i^{-1}\lambda_i(f)\lambda_i(g)=0. \tag{7}$$

Proof. Let us first observe that $\Lambda\sim\mu$ if and only if

$$\ker\Lambda''=\ker\mu'', \tag{8}$$

$$\sum_{i=1}^{m} b_i^{-1}\lambda_i(f)\lambda_i(g)=\sum_{i=1}^{m} b_i^{-1}\mu_i(f)\mu_i(g) \quad \text{for } f,g\in H_\xi^A. \tag{9}$$

Now the proof of the only if part is obvious. In order to prove the if part let us denote for each $t\in I$ by B_t^+ (B_t^-) the subspace of those $f\in H^n(I)$ for which $\operatorname{supp} f\subset[0,t]$ $(\operatorname{supp} f\subset[t,1])$. Let then $C_t^\pm=\Lambda''(B_t^\pm)$. Obviously C_t^+, $t\in I$ is a monotonic family of subspaces of R^k $k=n-m$ and as such can have jumps in at most p points, $p\leqq k$, say $s_1,\ldots,s_p$ of I. Assume that $0\leqq s_1<\ldots<s_p\leqq 1$. The point $s_1=0$ iff $C_t^+\neq 0$ for all $t>0$ and $s_p=1$ iff $C_t^+\neq R^k$ for all $t<1$. From (6) it follows that for $t\in I$ the spaces C_t^+ and C_t^- are lineary independent. Further for

$x \in R^k$ let $f \in H^n(I)$ be such that $\Lambda'' f = x$. For each $t < t'$ the function f can be decomposed into $f = g + h$ with $g \in B_t^+$ and $h \in B_t^-$. But if $t \notin \{s_1, \ldots, s_p\}$ there exists $t' > t$ such that $C_t^+ = C_{t'}^+$ and we obtain $x = \Lambda'' f = \Lambda'' g + \Lambda'' h$ with $\Lambda'' g \in C_{t'}^+ = C_t^+$ and $\Lambda'' h \in C_t^-$. This proves that $R^k = \operatorname{span}\{C_t^+, C_t^-\}$ for $t \notin \{s_1, \ldots, s_p\}$ hence for such t, R^k is a direct sum of C_t^+ and C_t^-. Consequently the family C_t^-, $t \in I$ can have jumps at the same points $s_1, \ldots, s_p$ only. Thus we obtain p pairs of complementary subspaces C_i^+, C_i^-, $1 \leqq i \leqq p$ where we have denoted $C_i^\pm = C_t^\pm$ for $s_i < t < s_{i+1}$, $i = 1, \ldots, p-1$, $C_p^+ = R^k$. Let us take such a scalar product in R^k denoted by $(\ ,\ )$ which makes each C_i^+ orthogonal to C_i^-. We have

$$R^k = \bigoplus_{i=1}^{p} E_i \quad \text{where } E_1 = C_1^+ \text{ and } E_i = C_i^+ \ominus C_{i-1}^+ \text{ for } 1 < i \leqq p.$$

Let $e_1, \ldots, e_k$ be the orthogonal with respect to $(\ ,\ )$ basis in R^k such that $\{e_j, j \in \alpha_i\}$ is a basis in E_i, $1 \leqq i \leqq p$. We define $\mu'' = (\mu_i, m < i \leqq n)$ by the formulae

$$\mu_{i+m}(f) = (e_i, \Lambda'' f) \qquad f \in H^n(I),\ i = 1, \ldots, n-m. \tag{10}$$

Obviously $\ker \mu'' = \ker \Lambda''$. We prove now that for $j \in \alpha_i$ we have $\operatorname{supp} \mu_j = \{s_i\}$. Indeed if $f \in H^n(I)$ is such that $f(t) = 0$ for t in some vicinity of s_i then we have $f = g + h$ with $g \in B_t^+$ for some $t < s_i$ and $h \in B_t^-$ for some $t > s_i$ hence $\Lambda'' g \in C_{i-1}^+$ and $\Lambda'' h \in C_i^-$. Since E_i is orthogonal to C_{i-1}^+ and C_i^- we have $(e_j, \Lambda'' g) = (e_j, \Lambda'' h) = 0$. This proves that μ_j vanishes outside of an arbitrary neighbourhood of s_i and thus $\operatorname{supp} \mu_j = \{s_i\}$ for $j \in \alpha_i$. Taking $t_{j+m} = s_i$ for $j \in \alpha_i$ we obtain $(t_{m+1}, \ldots, t_n)$ and $\mu'' = (\mu_{m+1}, \ldots, \mu_n)$ such that (8) is satisfied.

In order to construct $\mu' = (\mu_1, \ldots, \mu_m)$ we proceed in a similar way. Let now $\tilde{B}_t^- = \{f \in \ker \Lambda'';\ \operatorname{supp} f \subset [t, 1]\}$, $\tilde{B}_t^+ = \{f \in \ker \Lambda'';\ \operatorname{supp} f \subset [0, t]\}$ and $\tilde{C}_t^\pm = \Lambda' \tilde{B}_t^\pm$. Since $\ker \mu'' = \ker \Lambda''$ and $\operatorname{supp} \mu_i = \{t_i\}$ (a one point set), by a similar argument as before, we have $R^m = \operatorname{span}\{\tilde{C}_t^+, \tilde{C}_t^-\}$ whenever $t \neq t_i$, $1 \leqq i \leqq p \leqq m$ for some sequence $0 \leqq t_1 < \ldots < t_p \leqq 1$. Condition (7) implies that $\tilde{C}_t^+$ is orthogonal to $\tilde{C}_t^-$ in the scalar product in R^m given by $(x, y)_b = \sum_{i=1}^{m} b_i^{-1} x_i y_i$. Let us now define $\mu' = (\mu_1, \ldots, \mu_m)$ by the formulae

$$\mu_i(f) = b_i^\pm (d_i, \Lambda' f)_b \qquad f \in \ker \Lambda'',\ 1 \leqq i \leqq m$$

where $(d_i, 1 \leqq i \leqq m)$ is a basis in R^m orthonormal with respect to $(\ ,\ )_b$ and such that $(d_j, j \in \tilde{\alpha}_i)$ is a basis in $\tilde{C}_i^+$ where $\tilde{\alpha}_i$, $\tilde{C}_i^+$ are defined in an analogous way as α_i and C_i^+. Again we have for $j \in \tilde{\alpha}_i$, if $f \in \ker \Lambda''$ and $f(t) = 0$ in any neighbourhood of t_i, then $\mu_j(f) = 0$. Such a functional can now be extended to the whole $H^n(I)$ in such a way that $\operatorname{supp} \mu_j = \{t_i\}$ for $j \in \alpha_i$. Moreover for $f, g \in \ker \Lambda'' = H_\xi^\Lambda$ we have

$$\sum_{i=1}^{m} b_i^{-1} \lambda_i(f) \lambda_i(g) = (\Lambda' f, \Lambda' g)_b = \sum_{i=1}^{m} (d_i, \Lambda' f)_b (d_i, \Lambda' g)_b$$

$$= \sum_{i=1}^{m} b_i^{-1} \mu_i(f) \mu_i(g)$$

and thus (9) is proved which together with (8) gives us that $\Lambda \sim \mu$.

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Changements de temps de processus de markov

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Introduction

L'objet de ce travail est l'étude des processus changés de temps d'un processus fortement markovien.

Si C_t est un processus croissant adapté à une filtration $\mathbf{F}_t$ sur Ω satisfaisant aux conditions habituelles, on définit l'inverse à droite j de C par $j_t(\omega) = \inf\{u | C_u(\omega) > t\}$. On construit ainsi une nouvelle filtration $\bar{F}_t = \mathbf{F}_{j_t}$ qui est encore continue à droite et complète.

En particulier si $(\Omega, \mathbf{F}_t, X_t, \theta_t, (P^x) x \in E)$ est un processus fortement markovien et C une fonctionnelle additive de ce processus on sait que le processus changé de temps $(\Omega, \bar{\mathbf{F}}_t, (X_{j_t}), (\theta_{j_t}), (P^x) x \in E)$ est encore fortement markovien. Par contre si C n'est pas continue et si T désigne un saut de cette fonctionnelle le processus changé de temps $\bar{X}_t(.) = X_{j_t}(.)$ est constant sur l'intervalle $[C_T(.), (C_T + S_T C_T)(.)[$.

Il s'ensuit que ce processus est non homogène. Ceci conduit a introduire un processus a deux composantes dont la prmière est linéaire sur les intervalles de type précédent, ce qui assurera le caractère markovien du processus. Plus précisement on définit sur:

$\hat{\Omega} = \mathbb{R}_+ \times \Omega$ un semi groupe $(\hat{\theta}_t)$ d'opèrateurs de translation en posant

$$\hat{\theta}_t(\lambda, \omega) = (\bar{R}_{t-\lambda}(\omega), \theta_{j_{t-\lambda}}(\omega)) \quad \text{où} \quad \bar{R}_{t-\lambda}(\omega) = C_{j_{t-\lambda}}(\omega) - (t - \lambda).$$

Définissant alors $\hat{X}_t$ par $\hat{X}_t(\lambda, \omega) = (\bar{R}_{t-\lambda}(\omega), X_{j_{t-\lambda}}(\omega))$; on établit que le processus $(\hat{\Omega}, \hat{\mathbf{F}}_t, \hat{\theta}_t, \hat{X}_t, \hat{P}^{(\lambda, x)} = (\varepsilon_\lambda \otimes P^x))$ est fortement markovien, $(\hat{\mathbf{F}}_t)_{t \in \mathbb{R}_+}$ étant une famille de tribu satisfaisant aux conditions habituelles et contenant la filtration naturelle de $\hat{X}$.

En fait ce résultat ne dépend que du comportement de X et C sur l'ensemble aléatoire $\bar{M}$ des points de croissance de C, et les hypothèses minimales pour sa validité s'expriment en terme de systèmes régénératifs.

Nous étudions ensuite le «système de Levy» de X.

Deux raisons nous incitent à en faire une étude directe: d'une part nous ne connaissons pas de conditions générales sous lesquelles $\hat{X}$ soit un processus

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droit et surtout nous nous proposons de calculer des projections duales prévisibles relatives à la filtration $(\hat{\mathbf{F}}_t)$.

La première étape consiste en la description des $\hat{\mathbf{F}}$ temps d'arrêts totalement inaccessibles. Ceux-ci se déduisent aisément des $\bar{\mathbf{F}}$ temps d'arrêts totalement inaccessibles qui eux-mêmes se décomposent en trois types. Un premier type est lié aux temps de saut du processus initial X que l'on suppose droit. Un second est lié aux sauts de la fonctionnelle additive C; tandis que la troisième classe est en relation avec l'ensemble M des extrémités gauche des intervalles contigus à l'ensemble aléatoire $\bar{M}$.

Soient f une fonction borélienne positive définie sur $(\mathbb{R}_+ \times E - \bar{B})^2$ nulle sur la diagonale et $\hat{K}_t = \sum_{s \leqq t} f(\hat{X}_{s-}, \hat{X}_s)$, la sommation portant sur l'ensemble aléatoire réunion des graphes des $\hat{\mathbf{F}}$ temps d'arrêts totalement inaccessibles. On montre que le calcul de la projection $\hat{\mathbf{F}}$ duale prévisible de $\hat{K}$: ${}^{p}\hat{K}$ se ramène en fait grâce a un résultat de [7] à celui de la projection $\mathbf{F}$ duale prévisible d'un processus croissant K.

La description précédemment esquissée des $\bar{\mathbf{F}}$ temps d'arrêts totalement inaccessibles permet alors de décomposer K_t en somme de processus croissants dont on sait calculer les projections duales en utilisant les théories du système de Levy et du balayage.

I. Notations et préliminaires

1) Définitions

Soit E un sous espace borélien d'un espace métrique compact. On distingue dans E un point cimetière δ. On désignera par $\mathscr{E}$ la tribu des ensembles boreliens de E et par $\mathscr{E}^*$ celle des ensembles universellement mesurables.

Ω désigne l'ensemble des applications continues à droite de $\mathbb{R}_+$ dans E qui conservent la valeur δ après leur temps d'entrée dans $\{\delta\}$ X_t et Θ_t désignent les applications coordonnées et les opérateurs de translations: pour tous $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$ et $\omega \in \Omega$, on a par définition:

$$X_s(\omega) = \omega(s), \qquad X_s(\Theta_t(\omega)) = X_{s+t}(\omega).$$

On pose $X_\infty(\omega) = \delta$, $\Theta_\infty(\omega) = [\delta]$, l'application constante sur δ.

$\mathbf{F}^0_\infty$ (resp. $\mathbf{F}^0_t$) désigne la tribu sur Ω engendrée par les variables $X_s, s \geqq 0$ (resp. $X_s, 0 \leqq s \leqq t$).

Si P est une probabilité sur $(\Omega, \mathbf{F}^0_\infty)$, $\mathbf{F}^P_\infty$ désigne la tribu P complétée de $\mathbf{F}^0_\infty$ et F^P_t la tribu obtenue par adjonction des ensembles P négligeables de F^P_∞ à $\mathbf{F}^\circ_t + = \bigcap_{\varepsilon > 0} F^0_{t+\varepsilon}$.

D'autre part on se donne pour tout $x \in E$ une probabilité p^x sur $(\Omega, \mathbf{F}^0)$ et on suppose que $p^\delta = \varepsilon_{[\delta]}$ et que l'application $x \to p^x(A)$ est $\mathscr{E}^*$-mesurable pour tout $A \in \mathbf{F}^0$. Si $\mathfrak{N}$ désigne l'ensemble des lois de probabilité sur $(E, \mathscr{E})$, pour toute μ de $\mathfrak{N}$ on définit la probabilité $p^\mu = \int \mu(sx)\, p^x$ par intégration et on pose

$$\mathbf{F}^\mu = \mathbf{F}^{p^\mu}_\infty, \qquad \mathbf{F}^\mu_t = \mathbf{F}^{p^\mu}_t, \qquad \mathbf{F} = \bigcap_{\mu \in \mathfrak{N}} \mathbf{F}^\mu, \qquad \mathbf{F}_t = \bigcap_{\mu \in \mathfrak{N}} \mathbf{F}^\mu_t.$$

La famille $(\mathbf{F}_t)_{t\in\mathbb{R}_+}$ est une famille croissante continue à droite de sous tribus de $\mathbf{F}$. Pour tout t et tout μ de $\mathfrak{N}$, F_t contient les ensembles μ négligeables de F.

On désignera par ξ le temps de mort du processus X:

$$\xi(\omega)=\mathrm{Inf}\{t\geqq 0, X_t(\omega)\in[\delta]\}$$

ξ est un $\mathbf{F}$ temps d'arrêt et $X_\xi\equiv\delta$.

Soit $(C_t)_{t\in\mathbb{R}_+}$ un processus croissant, continu à droite, $(\mathbf{F}_t)$ adapté. On n'exige pas que $C_0\equiv 0$, mais par contre on supposera que

$$C_\infty=\lim_{t\to\infty} C_t=C_\xi. \tag{1}$$

On désigne par $\overline{M}$ l'ensemble fermé aléatoire des points de croissance de C et par M le fermé droit aléatoire minimal d'adhérence $\overline{M}$.

Tout point de M isolé à droite dans M est aussi isolé à gauche dans M.

Nous définirons enfin les fonctions aléatoires suivantes:

$$D_t(\omega)=\inf\{s|s>t,(s,\omega)\in\overline{M}\}=\inf\{s|s>t,(s,\omega)\in M\} \tag{2}$$

(on convient que $\inf(\emptyset)=+\infty$)

$$L_t(\omega)=\sup\{s|0\leqq s\leqq t,(s,\omega)\in\overline{M}\}=\sup\{s|0\leqq s\leqq t,(s,\omega)\in M\} \tag{3}$$

(avec la convention usuelle $\sup[\emptyset]=0$).

$$\ell_t(\omega)=\sup\{s|0\leqq s<t,(s,\omega)\in\overline{M}\}. \tag{4}$$

Nous allons maintenant rappeler certains résultats sur les changements de temps ([7]).

2) *Changements de temps*

Nous nous placerons dans le cadre décrit précédemment.

Soit

$$z=\inf\{t|C_t=+\infty\})$$

z est un $\mathbf{F}$ temps d'arrêt vérifiant $C_z=C_\infty$ et en vertu de la relation (1) on a $z\leqq\xi$.

Désignons par j et i les inverses à droite et à gauche de C définis par:

$$\begin{cases} j_t \equiv\inf\{s|C_s>t\} & \text{si } t<\infty \\ j_\infty=z \end{cases} \tag{5}$$

$$\begin{cases} i_t =\sup\{s|C_s<t\} & \text{si } t>0 \\ i_0=0 \end{cases} \tag{6}$$

La relation suivante sera d'un usage courant:

$$a\leqq C_b\Leftrightarrow i_a\leqq b \quad (a\in\mathbb{R}_+,\ b\in\overline{\mathbb{R}}_+). \tag{7}$$

Pour tout t, j_t et i_t sont des $\mathbf{F}$ temps d'arrêts; ce qui nous permet en posant $\bar{\mathbf{F}}_t = \mathbf{F}_{j_t}$ de définir une filtration $(\bar{\mathbf{F}}_t)$ continue à droite et complète telle que $\bar{\mathbf{F}}_\infty = \mathbf{F}_z$.

(j_t) est alors un processus croissant $\bar{\mathbf{F}}$ adapté dont les inverses à droite et à gauche sont C et B.

$$C_t = \inf\{s | j_s > t\}; \quad B_t = \sup\{s | j_s < t\}.$$

On vérifie que $B_t = C_{t-} = \sup_{s<t} C_s$, ce qui implique par symétrie que $i_t = j_{t-}$.

De même si $\bar{z} = \inf\{t | j_t = +\infty\}$, $\bar{z}$ est un $\bar{\mathbf{F}}$ temps d'arrêt satisfaisant à $j_{\bar{z}} = j_\infty = z$, $\bar{z} = B_\infty = C_\infty$.

Considérons enfin les composés des processus précédemment définis:

$$\begin{aligned} D'_t &= j_{C_t} & \bar{D}'_t &= C_{j_t} \\ \ell'_t &= i_{B_t} & \bar{\ell}'_t &= B_{i_t} \\ R_t &= (D'_t - t \wedge z) & \bar{R}_t &= \bar{D}'_t - t \wedge \bar{z}. \end{aligned} \tag{8}$$

Notons que ces processus sont liés aux fonctions D_t et ℓ_t précedemment définie par les relations

$$\begin{cases} D'_t = D_t \wedge z \\ \ell'_t = \ell_t. \end{cases} \tag{9}$$

Enfin on a les relations:

$$-) \; z \geqq D'_t \geqq t \wedge z \quad \text{et} \quad 1_{\{D'_t > t \wedge z\}} = 1_{\{z \geqq D_t > t\}} = \sum_{s \in \mathbb{R}_+} 1_{\{i_s \leqq t < j_s\}}$$

$$-) \; \bar{z} \geqq \bar{D}'_t \geqq t \wedge \bar{z} \quad \text{et} \quad 1_{\{\bar{D}_t > t \wedge \bar{z}\}} = \sum_{s \in \mathbb{R}_+} 1_{\{B_s \leqq t < C_s\}}$$

$$-) \; \ell'_t \leqq t \wedge z \quad \text{et} \quad 1_{\{\ell'_t < t \wedge z\}} = \sum_{s \in \mathbb{R}_+} 1_{\{i_s < t \leqq j_s\}}$$

$$-) \; \bar{\ell}'_t \leqq t \wedge \bar{z} \quad \text{et} \quad 1_{\{\bar{\ell}'_t < t \wedge \bar{z}\}} = \sum_{s \in \mathbb{R}_+} 1_{\{B_s < t \leqq C_s\}}.$$

Pour tout t, ℓ_t (respectivement $\bar{\ell}_t$) est un $\mathbf{F}$ (respectivement $\bar{\mathbf{F}}$) temps d'arrêt, et le processus ℓ_t (resp. $\bar{\ell}_t$) est $\mathbf{F}$ (resp. $\bar{\mathbf{F}}$) prévisible.

L'ensemble aléatoire $G = \{\ell_t = t \wedge Z\}$ (resp. $\bar{G} = \{\bar{\ell}_t = t \wedge \bar{Z}\}$) est $\mathbf{F}$ (resp. $\bar{\mathbf{F}}$) prévisible et ces ensembles sont liés par les relations:

$$\begin{aligned} t \wedge z \in G &\Rightarrow B_{t \wedge z} \in \bar{G} \\ t \wedge \bar{z} \in \bar{G} &\Rightarrow i_{t \wedge \bar{z}} \in G. \end{aligned} \tag{11}$$

Notons que G représente l'ensemble des points d'accumulation à gauche de points de $\bar{M}$ auquel on adjoint (éventuellement) le graphe du temps d'arrêt z.

Pour tout t, D'_t (resp. $\bar{D}'_t$) est un $\mathbf{F}$ (resp. $\bar{\mathbf{F}}$) temps d'arrêt, mais processus D'_t n'est pas $(\mathbf{F})$ adapté, alors que $\bar{D}'_t$ est $(\bar{\mathbf{F}})$ adapté.

L'ensemble $R = \{D'_t = t \wedge z\}$ est un ensemble $\mathbf{F}$ progressivement mesurable lié à l'ensemble $\bar{\mathbf{F}}$ optionnel $\bar{R} = \{\bar{D}'_t = t \wedge \bar{z}\}$ par les relations:

$$\begin{aligned} t \in R &\Rightarrow C_t \in \bar{R} \\ t \in \bar{R} &\Rightarrow j_t \in R. \end{aligned} \tag{12}$$

Remarquons encore que R est réunion de l'ensemble aléatoire des points d'accumulation à droite de points de M (ou de $\bar{M}$) et du graphe du temps d'arrêt z.

Rappelons enfin les résultats de théorie générale que nous utiliserons:

R.1.: Si T est un $\bar{\mathbf{F}}$ temps d'arrêt inférieur à $\bar{z}$, j_T est un $\mathbf{F}$ temps d'arrêt et les tribus $\bar{\mathbf{F}}_T$ et $\mathbf{F}_{j_T}$ coïncident ([7] proposition 2 et corollaire 7).

R.2.: Un $\bar{\mathbf{F}}$ temps d'arrêt T inférieur à $\bar{z}$ est totalement inaccessible si et seulement si:

a) le graphe de la variable aléatoire i_T est disjoint de tout graphe de $\mathbf{F}$ temps d'arrêt prévisible.

b) le graphe de T est presque sûrement inclus dans $\bar{G}$ ([7] corollaire 5).

R.3.: Si $\bar{\kappa}_t$ est un processus croissant satisfaisant à $\bar{\kappa}_{\bar{z}-}=\bar{\kappa}_\infty$ et $\bar{\kappa}_0=0$, et dont le support est inclus dans $\bar{G}$, sa projection duale $\bar{\mathbf{F}}$ prévisible ${}^p\bar{\kappa}_t$ est égale à ${}^p\kappa_{j_t}$, où ${}^p\kappa$ est la projection duale $\mathbf{F}$ prévisible du processus croissant $\kappa_t=\bar{\kappa}_{C_t}$ ([7] théorème 9).

3) Présentation des hypothèses et résultats préliminaires

On sait ([5]) que si X est un processus fortement markovien et C une fonctionnelle additive continue le processus $\bar{X}_t=X_{j_t}$ est encore fortement markovien. Mais on peut voir que cette propriété ne dépend en fait que du comportement de X et C sur l'ensemble $\bar{M}$. Nous allons préciser ce point en introduisant les hypothèses minimales pour la solution de ce problème dans le cas où C est discontinue.

Posons

$$\Gamma=\{(t,\omega)|\, C_t(\omega)>B_t(\omega)\},$$

Γ est une réunion dénombrable de graphes de temps d'arrêts. D'autre part le fermé droit minimal M d'adhérence $\bar{M}$ peut également s'écrire comme réunion des graphes des $\mathbf{F}$ temps d'arrêts $D_t(t\in\eta\mathbb{R}_+)$.

Soient

$$\hat{M}=M\cup\Gamma \quad \text{et} \quad \tilde{M}=\hat{M}\cup[[0]].$$

Nous les deux hypothèses suivantes:

H.1.: ($\hat{M}$ additivité): Pour toute variable aléatoire positive S et tout $\mathbf{F}$ temps d'arrêt T dont le graphe est inclus dans $\hat{M}$, on a: pour tout ω de

$$\Omega\; C_{S(\omega)+T(\omega)}(\omega)=C_T(\omega)+C_{s(\omega)}(\Theta_T(\omega)). \tag{13}$$

H.2.: ($\tilde{M}$ regenération): Pour tout $\mathbf{F}$ temps d'arrêt T dont le graphe est inclus dans $\tilde{M}$ et pour tout B de $\mathbf{F}$ on a:

$$E^\mu[\Theta_T^{-1}(B)|\mathbf{F}_T]=P^{X_T}(B) \quad P^\mu \text{ps pour toute } \mu \text{ de } \mathfrak{M}. \tag{14}$$

Exemple: $\bar{M}$ est un fermé aléatoire homogène M est le fermé droit minimal associé. Le processus croissant $L_t=\sup\{s|s\leqq t, s\in M\}$ vérifie la relation $(L_{s+t}-t)^+=L_s\circ\Theta_t$.

L'ensemble des sauts de L est inclus dans M. L vérifie la propriété H_1, mais n'est pas une fonctionnelle additive.

Quant à la propriété H_2, elle est dans ce cas équivalente à la propriété de régénération de [10].

Revenons au cas général, nous commencerons par établir le résultat suivant:

Lemme 1. *Soit T un $\bar{\mathbf{F}}$ temps d'arrêt.*

a) *le graphe du* $\mathbf{F}$ *temps d'arrêt j_T est inclus dans $\hat{M}$.*

b) *Si $B\in\mathbf{F}$, $E^{\mu}(\Theta_{j_T}^{-1}(B)|\bar{\mathbf{F}}_T)=P^{X_{j_T}}(B)$ P^{μ} p.s. pour toute μ de $\mathfrak{N}$.*

Démonstration. Soit $\omega\in\Omega$ et $t=j_T(\omega)$ alors

$$t=t\,1_{\{C_t=B_t\}}\,1_M(t)+t\,1_{\{C_t>B_t\}}\,1_M(t)$$
$$=D_t\,1_{\{C_t=B_t\}}+t\,1_\Gamma(t).$$

et a) résulte de la rélation $\hat{M}=\bigcup_{t\geqq 0}[[D_t]]\cup\Gamma$. Quant à b) il résulte de H_2, de a) et de la relation $\mathbf{F}_{j_T}=\bar{\mathbf{F}}_T$ (R.1.).

Le lemme 1 permet de mettre en évidence une propriété «presque markovienne» du système $(\Omega,(\bar{\mathbf{F}}_t),(X_{j_t}),\ldots)$. De même le résultat suivant va établir que j est «presque» une $\bar{\mathbf{F}}$ fonctionnelle additive.

Nous ferons dorénavant les conventions suivantes:

Si $a<0$ $j_a=0$ et $\bar{R}_a=-a$.
On peut alors établir le résultat suivant:

Lemme 2. *Si S est une variable aléatoire* $\mathbf{F}$ *measurable positive et T un $\bar{\mathbf{F}}$ temps d'arrêt, on a:*

$$j_{S+T}(\omega)=j_T(\omega)+j_{(S-\bar{R}_T)(\omega)}(\theta_{j_T}(\omega)) \tag{15}$$

pour tout ω de Ω.

Démonstration. – Sur $\{S<\bar{R}_T\}$ on a $T\leqq S+T<C_{j_T}$. Par suite $j_{S+T}=j_T$ alors que $j_{(S-\bar{R}_T)(\omega)}\equiv 0$.

– Sur $\{S-\bar{R}_T\geqq 0\}$ on a

$$j_{(S-\bar{R}_T)(\omega)}(\theta_{j_T}(\omega))=\mathrm{Inf}\{u|\,C_u(\theta_{j_T}(\omega))>(S-\bar{R}_T)(\omega)\}.$$

Puisque d'après le lemme 1 $[[j_T]]\subseteq\hat{M}$, il résulte de H.1 que:

$$C_u\circ\theta_{j_T}(\omega)=C_{u+j_T}(\omega)-C_{j_T}(\omega).$$

Par suite on a:

$$j_{(S-\bar{R}_T)(\omega)}(\theta_{j_T}(\omega))=\mathrm{Inf}\{u|\,C_{u+j_T}(\omega)>(S+T)(\omega)\}=j_{S+T}(\omega)-j_T(\omega).$$

Il résulte de la relation (15) que, contrairement au cas où C est continue, le système (X_{j_t},θ_{j_t}) ne permet pas de définir un semi groupe d'opérateurs de translation et un processus homogène sur Ω.

Nous allons, en nous inspirant d'une méthode utilisée dans [10] construire dans le prochain paragraphe une réalisation dans un «espace plus grand» du processus changé de temps qui sera fortement markovienne.

II. Construction d'un processus de Markov

Posons $\hat{\Omega}=\overline{\mathbb{R}}_+\times\Omega$ et définissons sur $\hat{\Omega}$ les opérateurs $(\hat{\theta}_t)_{t\in\mathbb{R}_+}$ par:

$$\hat{\theta}_t(\lambda,\omega)=\begin{cases}(\lambda-t,\omega) & \text{si } \lambda>t\\ (C_{j_{t-\lambda}}(\omega)-(t-\lambda),\theta_{j_{t-\lambda}}(\omega)) & \text{si } t\geqq\lambda\end{cases}$$

Notons qu'avec les conventions faites précédemment:

$$\hat{\theta}_t(\lambda,\omega)=(\overline{R}_{t-\lambda}(\omega),\theta_{j_{t-\lambda}}(\omega)).$$

On a alors:

Lemme 3. *$\hat{\theta}_{s+t}=\hat{\theta}_t(\hat{\theta}_s)$ pour tous $(s,t)\in\mathbb{R}_+\times\mathbb{R}_+$.*

La démonstration de ce lemme est routinière et sera omise.

Définition de $\hat{X}_t$. On pose $\hat{X}_t(\lambda,\omega)=(\overline{R}_{t-\lambda}(\omega),X_{j_{t-\lambda}}(\omega))$. Notons que si $\lambda>t$ on a $\hat{X}_t(\lambda,\omega)=(\lambda-t,X_0(\omega))$ et qu'en particulier:

$$\hat{X}_0(\lambda,\omega)=\begin{cases}(\lambda,X_0(\omega)) & \text{si } \lambda>0\\ (C_{j_0}(\omega),X_{j_0}(\omega)) & \text{si } \lambda=0\end{cases}$$

$\hat{X}$ est à valeurs $\hat{E}=\overline{\mathbb{R}}_+\times E$ que l'on munit de la topologie produit.

On distingue dans $\hat{E}$ le point cimetière $\hat{\delta}=(+\infty,\delta)$ et on pose $\hat{X}_\infty(\hat{\omega})=\hat{\delta}$ et $\hat{\theta}_\infty(\hat{\omega})=[\hat{\delta}]$; $\hat{\omega}\in\hat{\Omega}$, ou $[\hat{\delta}]=(+\infty,\delta)$. On désignera par $\hat{\xi}(\hat{\omega})$ le temps d'entrée de $\hat{X}(\hat{\omega})$ dans $\{\hat{\delta}\}$. Notons que si $C_\infty(\omega)=C_\xi(\omega)=+\infty\,\xi(\hat{\omega})=+\infty$ et que si $C_\infty(\omega)=C_\xi(\omega)<\infty$ on aura $\hat{X}_t(\lambda,\omega)=\hat{\delta}$ si et seulement si $j_{t-\lambda}(\omega)\geqq z$ et $X_{j_{t-\lambda}}(\omega)=\delta$ soit, puisque dans ce cas $z\leqq\xi$.

$$\hat{X}_t(\lambda,\omega)=\hat{\delta}\Leftrightarrow j_{t-\lambda}(\omega)\geqq\xi(\omega)\Leftrightarrow t-\lambda\geqq C_\xi(\omega)=C_\infty(\omega).$$

Par suite dans tous les cas $\hat{\xi}(\lambda,\omega)=C_\infty(\omega)+\lambda$ et il est clair que si $s\geqq\hat{\xi}(\hat{\omega})\,\hat{X}_s(\hat{\omega})=\hat{\delta}$.

Enfin les trajectoires de $\hat{X}$ sont continues à droite.

De plus, nous avons facilement le résultat suivant:

Lemme 4. *$\hat{X}$ est $\hat{\theta}$ homogène: pour tous (s,t,λ,ω), $\hat{X}_{s+t}(\lambda,\omega)=X_s\circ\hat{\theta}_t(\lambda,\omega)$.*

Nous allons définir une filtration sur $\hat{\Omega}$. On peut naturellement considérer la filtration engendrée par $\hat{X}$, soit $\mathbf{G}_t^0=\sigma(\hat{X}_s,s\leqq t)$. Mais en procédant ainsi on ne tient pas compte du comportement de X en dehors de l'ensemble aléatoire $\overline{M}$.

Nous sommes donc amenés à introduire une famille $(\hat{\mathbf{F}}_t^0)_{t\in\mathbb{R}_+}$ de sous tribus de $(\mathscr{B}_{\mathbb{R}_+}\otimes\overline{\mathbf{F}}_\infty)$ de la façon suivante:

Convenons que si $U\subset\hat{\Omega}$ et si $\lambda\in\mathbb{R}_+$, U_λ désignera la section de $\hat{U}$ en λ: $\hat{U}_\lambda=\{\omega|(\lambda,\omega)\in\hat{U}\}$.

Posons alors:

$$\hat{\mathbf{F}}_t^0 = \{\hat{U} \mid \hat{U} \in \mathscr{B}_{\mathbb{R}_+} \otimes \bar{F}_\infty,\ \hat{U}_\lambda 1_{\{t \geq \lambda\}} \in \bar{\mathbf{F}}_{t-\lambda} \text{ et } \hat{U}_\lambda 1_{\{t<\lambda\}} \in \mathbf{F}_0\}$$

$(\hat{\mathbf{F}}_t^0)_{t \in \mathbb{R}_+}$ est une famille croissante continue à droite de sous tribus de $(\mathscr{B}_{\mathbb{R}_+} \otimes \bar{\mathbf{F}}_\infty)$ et $\hat{X}_t$ est $\hat{\mathbf{F}}_t^0$ adapté. Désignons alors par $\hat{\mathbf{F}}^0$ (resp. $\mathbf{G}^0$) la réunion des $\hat{\mathbf{F}}_t$: $\hat{\mathbf{F}}^0 = \bigcup_{t \in \mathbb{R}_+} \hat{F}_t^0$ (resp. $\mathbf{G}^0 = \bigcup_{t \in \mathbb{R}_+} G_t^0$) et par $\hat{\mathbf{F}}^*$ (resp. $\mathbf{G}^*$) la complétée universelle de $\hat{\mathbf{F}}^0$ (resp. $\mathbf{G}^0$).

Si $(\lambda, x) \in \hat{E}$ on pose $\hat{P}^{(\lambda,x)} = \varepsilon_\lambda \otimes P^x$ et si $\hat{\mu}$ est une loi de probabilité sur $\hat{E}$ (ce que l'on notera $\hat{\mu} \in \hat{\mathfrak{N}}$), $\hat{P}^{\hat{\mu}}$ est défini par intégration: $\hat{P}^{\hat{\mu}} = \int \hat{P}^{(\lambda,x)}\, d\hat{\mu}(\lambda, x)$. Soit alors $\mathscr{N} = \bigcap_{\hat{\mu} \in \hat{\mathfrak{N}}} \mathscr{N}_{\hat{\mu}}$ où $\hat{\mathscr{N}}^{\hat{\mu}} = \{\hat{U} \in \hat{\mathbf{F}}^*, \hat{P}^{\hat{\mu}}(\hat{U}) = 0\}$.

Finalement on définit $\hat{\mathbf{F}}_t = \hat{\mathbf{F}}_t^0 \vee \hat{\mathscr{N}}$, $\mathbf{G}_t = \mathbf{G}_t^0 \vee \hat{\mathscr{N}}$, $\hat{\mathbf{F}} = \vee \hat{\mathbf{F}}_t$ $\mathbf{G} = \vee \mathbf{G}_t$.

$(\hat{\mathbf{F}}_t)$ (resp. $\mathbf{G}_t$) est une famille continue à droite de sous tribus de $\hat{F}\, \hat{P}^{\mu}$ complète pour toute loi $\hat{\mu} \in \hat{\mathfrak{N}}$.

Nous commencerons par remarquer que les propriétés relatives au système $(\hat{\Omega}, (\hat{\mathbf{F}}_t), \hat{P}^{(\lambda,x)})$ se déduisent de celles relatives au système $(\Omega, (\bar{\mathbf{F}}_t), P^x)$. A cet effet nous introduirons les notations suivantes:

Notations. – Si $\hat{T}(\lambda, \omega)$ est un fonction positive définie sur $\hat{\Omega}$ on pose $\hat{T}_\lambda(\omega) = (\hat{T}(\lambda, \omega) - \lambda) \vee 0$.

– Si $\hat{Z}_t(\lambda, \omega)$ est un processus sur $\hat{\Omega}$ on définit un processus $\hat{Z}^\lambda$ sur Ω par $\hat{Z}_t^\lambda(\omega) = \hat{Z}_{t+\lambda}(\lambda, \omega)$.

– Si $\bar{T}$ est une fonction positive sur Ω on pose $\hat{T}(\lambda, \omega) = \bar{T}(\omega) + \lambda$.

– Si $\bar{Z}_\cdot$ est un processus sur Ω, on définit un processus $\hat{Z}$ sur $\hat{\Omega}$ par $\hat{Z}_t(\lambda, \omega) = \bar{Z}_{t-\lambda}(\omega) 1_{\{t > \lambda\}}$.

On a alors:

Proposition 5. a) *Si $\hat{T}$ est un $\hat{\mathbf{F}}$ temps d'arrêt (resp. prévisible), $\hat{T}_\lambda$ est pour tout $\lambda \in \mathbb{R}_+$ un $\bar{\mathbf{F}}$ temps d'arrêt (resp. prévisible).*

b) *Si $\bar{T}$ est un $\bar{\mathbf{F}}$ temps d'arrêt (resp. prévisible), $\hat{T}(\lambda, \omega) = \bar{T}(\omega) + \lambda$ est un $\hat{\mathbf{F}}$ temps d'arrêt (resp. prévisible).*

c) *$\hat{T}$ est un $\hat{\mathbf{F}}$ temps d'arrêt totalement inaccessible si et seulement si pour tout λ, $\hat{T}_\lambda$ est un $\bar{\mathbf{F}}$ temps d'arrêt totalement inaccessible.*

d) *Pour tout $\hat{\mathbf{F}}$ temps d'arrêt $\hat{T}$ on a:*

$$(\hat{\mathbf{F}}_{\hat{T}}^0)_\lambda \subseteq \bar{\mathbf{F}}_{\hat{T}_\lambda} \quad \text{où } (\hat{\mathbf{F}}_\cdot^0)_\lambda = \{\hat{u} \mid \hat{u} \in \hat{\mathbf{F}}_\cdot^0\}$$

e) *Sie $\hat{Z}$ est $\hat{\mathbf{F}}$ prévisible $\hat{Z}_\cdot^\lambda(\ ) = \hat{Z}(\lambda, .)$ est $\bar{\mathbf{F}}$ prévisible pour tout λ. Réciproquement si $\bar{Z}$ est $\bar{\mathbf{F}}$ prévisible, $\hat{Z}_t = \bar{Z}_{t-\lambda} 1_{\{t > \lambda\}}$ est $\hat{\mathbf{F}}$ prévisible.*

On peut grâce au rappel R.2. améliorer le résultat c) ci-dessus pour obtenir une caractérisation des $\hat{\mathbf{F}}$ temps d'arrêt totalement inaccessibles que nous utiliserons ultérieurement.

Corollaire 6. *Un $\hat{\mathbf{F}}$ temps d'arrêt $\hat{T}$ est totalement inaccessible si et seulement si:*

i) *$\hat{T}(\lambda, .) > \lambda$ pour tout réel positif λ*

ii) $[[\hat{T}_\lambda]] \subseteq \bar{G}$

iii) *Le graphe de la variable aléatoire $i_{\hat{T}_\lambda}$ est disjoint de tout graphe de $\mathbf{F}$ temps d'arrêt prévisible.*

Démonstration. Il suffit d'après ce qui précède d'établir a). Remarquons que l'ensemble $U=\{(\lambda,\omega)|\,\hat{T}(\lambda,\omega)\leqq\lambda\}$ est $\hat{\mathbf{F}}_{\hat{T}}$ mesurable. Par suite $\hat{T}_1$ défini par:

$$\hat{T}_1(\lambda,\omega)=\begin{cases}\hat{T}(\lambda,\omega) & \text{si } (\lambda,\omega)\in U\\ 0 & \text{sinon}\end{cases}$$

est un $\hat{\mathbf{F}}$ temps d'arrêt.

Il suffit maintenant de prouver que $\hat{T}_1$ est prévisible: Soit $t\in\mathbb{R}_+$ et

$$U^n=\left\{(\lambda,\omega)|\,\hat{T}_1(\lambda,\omega)\leqq\frac{n}{n-1}t\right\}$$

$$U^n_\lambda=\left\{\omega|\,\hat{T}_1(\lambda,\omega)\leqq\frac{n}{n-1}t\right\}$$

alors si

$$\frac{n}{n-1}t\geqq\lambda\ U^n_\lambda=\Omega\in\bar{F}_{t-\lambda}$$

et si

$$\frac{n}{n-1}t<\lambda\ U^n_\lambda\in F_0\subset\bar{F}_{t-\lambda}$$

$\frac{n-1}{n}\hat{T}_1$ est donc un $\hat{\mathbf{F}}$ temps d'arrêt et $\hat{T}_1$ est prévisible.

Nous allons maintenant établir le résultat essentiel de ce paragraphe:

Théorème 7. *Le processus* $(\hat{\Omega},\hat{F},\hat{F}_t,\hat{\theta}_t,\hat{X}_t,(\hat{P}^{\hat{\mu}})_{\hat{\mu}\in\mathfrak{M}})$ *est fortement markovien.*

Démonstration. Soient $\hat{T}$ un $\hat{\mathbf{F}}^0$ temps d'arrêt fini, $\hat{U}\in\hat{\mathbf{F}}^0_{\hat{T}}$, $(\lambda,x)\in\hat{E}$, $s\in\mathbb{R}_+$ et $f\in(\mathscr{B}_{\mathbb{R}_+}\otimes\mathscr{E})$. On veut prouver que:

$$\int 1_{\hat{U}}f(\hat{X}_{s+\hat{T}})\,d\hat{P}^{(\lambda,x)}=\int 1_{\hat{U}}\hat{E}^{\hat{X}_{\hat{T}}}[f(\hat{X}_s)]\,d\hat{P}^{(\lambda,x)}\tag{12}$$

or

$$\int 1_{\hat{U}}f(\hat{X}_{s+\hat{T}})\,d\hat{P}^{(\lambda,x)}=\int 1_{\hat{U}_\lambda}f(\hat{X}_{s+\hat{T}}(\lambda,.)\,dP^x$$

et de même

$$\int 1_{\hat{U}}\hat{E}^{\hat{X}_{\hat{T}}}[f(\hat{X}_s)]\,d\hat{P}^{(\lambda,x)}=\int 1_{\hat{U}_\lambda}\hat{E}^{\hat{X}_{\hat{T}}(\lambda,.)}(f(\hat{X}_s))\,dP^x.$$

Notons que $\hat{U}_\lambda=(\hat{U}_\lambda\cap\{\hat{T}(\lambda,.)<\lambda\})\cup(\hat{U}_\lambda\cap\{\hat{T}(\lambda,.)\geqq\lambda\})$ et reportons cette décomposition dans les relations précédentes. L'égalité (12) sera alors démontrée si l'on établi que

i) $\int 1_{\hat{U}_\lambda\cap\{\hat{T}(\lambda,.)<\lambda\}}\hat{E}^{\hat{X}_{\hat{T}}(\lambda,.)}(f(\hat{X}_s))\,dP^x=\int 1_{\hat{U}_\lambda\cap\{\hat{T}(\lambda,.)<\lambda\}}f(\hat{X}_{s+\hat{T}}(\lambda,.)\,dP^x$
et que

ii) $\int 1_{\hat{U}_\lambda\cap\{\hat{T}(\lambda,.)\geqq\lambda\}}\hat{E}^{\hat{X}_{\hat{T}}(\lambda,.)}(f(\hat{X}_s))\,dP^x=\int 1_{\hat{U}_\lambda\cap\{\hat{T}(\lambda,.)\geqq\lambda\}}f(\hat{X}_{s+\hat{T}}(\lambda,.)\,dP^x$.
Etablissons d'abord i).

Puisque sur $\{\hat{T}(\lambda,.)<\lambda\}$ on a $\hat{X}_{\hat{T}}(\lambda,.)=(\lambda-\hat{T}(\lambda,.),X_0(.))$; il s'ensuit que:

$$\hat{E}^{\hat{X}_{\hat{T}}(\lambda,\omega)}[f(\hat{X}_s)]=E^{X_0(\omega)}[f(\hat{X}_s(\lambda-\hat{T}(\lambda,\omega),\omega')]$$

et comme

$$\hat{X}_s(\lambda-\hat{T}(\lambda,\omega),\omega')=\begin{cases}(\lambda-s-\hat{T}(\lambda,\omega),X_0(\omega')) & \text{si } \lambda>s+\hat{T}(\lambda,\omega)\\ \hat{X}_s+\hat{T}(\lambda,\omega)^{(\lambda,\omega')} & \text{sinon}\end{cases}$$

Le membre de gauche de i) sera égal à la somme des deux termes suivants:

$$a_1 = \int 1_A(\omega)\, E^{X_0(\omega)}[f(\lambda - s - \hat{T}(\lambda,\omega), X_0(\omega')]\, dP^x(\omega).$$
$$a_2 = \int 1_B(\omega)\, E^{X_0(\omega)}[f(\hat{X}_{s+\hat{T}(\lambda,\omega)}(\lambda,\omega'))]\, dP^x(\omega)$$

où

$$A = \hat{U}_\lambda \cap \{\hat{T}(\lambda,.)+s<\lambda\} \in \mathbf{F}_0$$

et

$$B = \hat{U}_\lambda \cap \{\hat{T}(\lambda,.)<\lambda \leqq \hat{T}(\lambda,.)+s\} \in \mathbf{F}_0.$$

Appliquons donc la propriété de régénération en 0 du processus initial. On obtient:

$$a_1 = \int 1_A(\omega) f(\lambda - s - \hat{T}(\lambda,\omega), X_0(\omega))\, dP^x(\omega)$$

et

$$a_2 = \int 1_B(\omega) f(\hat{X}_{s+\hat{T}}(\lambda,\omega))\, dP^x(\omega).$$

Puisque sur $\{\hat{T}(\lambda,.)+s<\lambda\}$ on a $\hat{X}_{s+\hat{T}}(\lambda,.) = (\lambda - s - \hat{T}(\lambda,.), X_0(.))$ la relation (i) s'obtient en sommant les 2 égalités ci-dessus.

Pour établir la relation ii) notons que:

Sur

$$\{\hat{T}(\lambda,.) \geqq \lambda\}\ \hat{X}_{s+\hat{T}}(\lambda,\omega) = (\bar{R}_{s+\hat{T}_\lambda(\omega)}(\omega), X_{j_{s+\hat{T}_\lambda(\omega)}}(\omega)).$$

Posons $T = \hat{T}_\lambda$ c'est d'après la proposition 5 un $\bar{\mathbf{F}}$ t.a. et

$$A = \{\hat{U}_\lambda \cap \hat{T}(\lambda,.) \geqq \lambda\} \in \bar{\mathbf{F}}_T.$$

De plus il résulte du lemme 2 que:

$$(\bar{R}_{s+T}(\omega), X_{j_{s+T}}(\omega)) = (\bar{R}_{s-R_T(\omega)}, X_{j_{s-R_T(\omega)}}) \circ \theta_{j_T(\omega)},$$

si bien que

$$\int_A f(\hat{X}_{s+\hat{T}}(\lambda,\omega))\, dP^x(\omega) = \int_A f(\bar{R}_{s-R_T(\omega)}, X_{j_{s-R_T(\omega)}}) \circ \theta_{j_T(\omega)}\, dP^x(\omega)$$

Posons

$$\varphi(u,\omega) = f(\bar{R}_u(\omega), X_{j_u(\omega)})$$

$\varphi(u,\omega)$ est $\mathscr{B}_{[0,u]} \otimes \mathbf{F}_u$ mesurable et c'est un processus $\bar{\mathbf{F}}$ optionnel.

Si S est $\bar{\mathbf{F}}_T$ mesurable, la variable aléatoire. $(\omega,\omega') \to \varphi(S(\omega),\omega')$ est $\bar{\mathbf{F}}_T \otimes \bar{\mathbf{F}}_T$ mesurable et si $A \in \bar{\mathbf{F}}_T$ il résulte de la propriété régénérative (14) que:

$$\int_A \varphi(S(\omega), \theta_{j_T}(\omega))\, dP^x(\omega) = \int_A E^{X_{j_T(\omega)}}(\varphi(S(\omega),\omega'))\, dP^x(\omega).$$

Appliquons ceci à $S(\omega) = s - \bar{R}_T(\omega)$ on obtient que:

$$\int_A f(\hat{X}_{s+\hat{T}}(\lambda,\omega))\, dP^x(\omega) = \int_A E^{X_{j_T(\omega)}}(f(\bar{R}_{s-R_T(\omega)}(\omega'), X_{j_{s-R_T(\omega)}}(\omega'))\, dP^x(\omega)).$$

Or

$$(\bar{R}_{s-R_T(\omega)}(\omega'), X_{j_{s-R_T(\omega)}}(\omega')) = \hat{X}_s(\bar{R}_T(\omega), \omega')$$

et

$$E^{X_{j_T(\omega)}}[f(\bar{R}_{s-R_T(\omega)}(\omega'), X_{j_{s-R_T(\omega)}}(\omega')] = \hat{E}^{(R_T(\omega), X_{j_T(\omega)})}(f(\hat{X}_s)(\mu, \omega'))$$

et puisque sur $\{\hat{T}(\lambda, .) \geqq \lambda\}$

$$(\bar{R}_T(\omega), X_{j_T}(\omega)) = \hat{X}_{\hat{T}}(\lambda, \omega)$$

la relation ii) et le théorème sont maintenant établis.

On peut maintenant préciser le sime groupe et l'ensemble des points de branchement du processus $\hat{X}$.

Corollaire 8. (a) $(\hat{P}_t)_{t\in\mathbb{R}_+}$ *défini par* $\hat{P}_t((\lambda, x), f) = \hat{E}^{(\lambda,x)}[f(\hat{X}_t)]$ *est le semi groupe associé à* $\hat{X}$.

(b) *Si* $\hat{B}$ *(resp.* B*) désigne l'ensemble des points de branchement de* $\hat{X}$ *(resp.* X*). On a*

$$\hat{B} = \mathbb{R}_+ \times B \cup \{\{0\} \times F^c\}$$

où

$$F = \{x | x \in E, P^x(j_0 = 0) = 1\}$$

est l'ensemble des points réguliers de C.

Nous désignerons par $\hat{E}$ *l'espace* $\hat{E} - \hat{B}$.

Examinons maintenant un cas particulier du théorème 7:

Nous reprendrons l'exemple étudié en I.3.: $\bar{M}$ est un fermé aléatoire homogène, M le fermé droit minimal associé et le processus croissant considéré est $L_t = \sup\{s | s \in M \wedge]0, t]\}$. Ainsi qu'on l'a déjà remarqué les hypothèses H_1 et H_2 sont équivalentes à l'hypothèse de régénération de [10].

La construction précédente permet d'obtenir un processus

$$\hat{X} = (\mathbb{R}_+ \times \Omega, \hat{\mathbf{F}}^0, \hat{X}_t, \hat{P}^{(\lambda,x)})$$

où

$$\hat{X}_t(\lambda, \omega) = (D_{t-\lambda}(\omega) - (t - \lambda), X_{D_{t-\lambda}}(\omega))$$

et $U \in \hat{\mathbf{F}}_t^0$ si et seulement si

$$U_\lambda 1_{\{t \geqq \lambda\}} \in \mathbf{F}_{D_{t-\lambda}} \quad \text{et} \quad U_\lambda 1_{\{t < \lambda\}} \in \mathbf{F}_0.$$

Le théorème 7 nous assure que le processus $\hat{X}$ est fortement markovien. En fait, on peut montrer que cette propriété caractérise entièrement les systèmes régénératifs; c'est l'objet de la proposition suivante dont nous omettrons la démonstration.

Proposition 9. *Si le processus* $\hat{X}$ *est fortement markovien, le système* (X, M) *est régénératif.*

III. Etude des sauts de $\hat{X}$

Ce chapitre constitute un préliminaire à l'étude du système de Levy du processus $\hat{X}$ et vise à caractériser ses temps de saut totalement inaccessibles. Comme

nous ne connaissons pas de conditions satisfaisantes pour lesquelles $\hat{X}$ soit un processus droit nous allons étudier directement ses sauts. A cet effet nous ferons désormais les hypothèses suivantes:

$H'2$: *X est un processus droit*

$H'3$: $\overline{M}$ est un ensemble aléatoire homogène.

Nous commencerons par poser certaines notations et rappeler quelques conséquences des hypothèses $H'2$ et $H'3$.

Notations et rappels:

On désigne par:

τ: la topologie de E.

$\bar{\tau}$: la topologie de E résultant de la compactification de Ray.

$\hat{\tau}$: la topologie sur $\hat{E}$ produit de la topologie usuelle de $\mathbb{R}_+$ et de τ.

X_{t^-}: (resp. $X^*_{t^-}$, resp. $\hat{X}_{t^-}$) la limite à gauche (resp. lorsqu'elle existe) en t de X (resp. X, resp. $\hat{X}$) dans les topologies $\bar{\tau}$ (resp. τ, resp. $\hat{\tau}$)

$$\mathscr{S} = \{(t,\omega) | X_{t^-}(\omega) \neq X_t(\omega), X_{t^-}(\omega) \notin B\}.$$

Rappelons qu'un $\mathbf{F}$ temps d'arrêt T est totalement inaccessible si et seulement si son graphe est inclus dans $\mathscr{S}$ [8].

Rappelons maintenant les deux résultats suivants [1]:

R.4.: Il existe un système de Levy (N, L) constitué d'un noyau positif N sur E transformant les fonctions boréliennes en fonctions mesurables par rapport à la tribu engendrée par les fonctions excessives, tel que $N(x,\{x\})=0 \,\forall\, x \in E$ et d'une fonctionnelle additive continue L telle que $E^x[L_t] < \infty \,\forall\, t, \forall\, x$.

Le couple (N, L) jouit de la propriété suivante: pour toute fonction borélienne f, positive sur $E \times E$, le processus croissant $\sum_{0<s\leq t} f(X_{s^-}, X_s) 1_{\mathscr{S}}(s)$ a pour projection duale prévisible le processus $\int_0^t dL_s \int_E N(X_{s^-}, dy) f(X_{s^-}, y)$. ([1] théorème 1).

R.5.: Soit A une fonctionnelle additive $\mathbf{F}_t$ adaptée, purement discontinue et dont les sauts sont totalement inaccessibles. Il existe alors une fonction positive h sur $E \times E$, $\mathscr{E}^1 \otimes \mathscr{E}$ mesurable ($\mathscr{E}^1$ désigne la tribu des universellement mesurables) telle que A soit indistinguable de la fonctionnelle additive $\sum_{0<s\leq t} h(X_{s^-}, X_s) 1_{\mathscr{S}}(s)$. ([1] théorème 2).

Rappelons maintenant la décomposition suivante de l'ensemble aléatoire $\overline{M}$ que l'on peut trouver dans [11] ou dans [4].

Soit $M^{\rightarrow}$ l'ensemble des extrémités gauche des intervalles contingus à $\overline{M}$: $M^{\rightarrow} = \{t>0, L_t = t, D_t > t\}$.

F désignant toujours l'ensemble des points réguliers de C, c'est-à-dire $\{x | P^x(j_0=0)=1\}$, posons

$-)$ $\rho_F = \{(t,\omega), t>0, X_t(\omega) \in F\}$.

Notons que ρ_F est la projection optionnelle de l'ensemble $\{t=D_t\}$.

– $M_\pi^{\rightarrow} = M^{\rightarrow} \cap \rho_F$ alors $t \in M_\pi^{\rightarrow} \Leftrightarrow X_t \in F$, $X_t = X_{t-}$ il ne passe aucun graphe de **F** t *a* dans $M_\pi^{\rightarrow}$.(ps)

– $M_b^{\rightarrow} = M^{\rightarrow} \cap \rho_F^c$

– $M_s^{\rightarrow} = M_b^{\rightarrow} \cap \{(t,\omega),\ X_{t-}(\omega) \notin B,\ X_t \neq X_{t-}\}$

– $M_a^{\rightarrow} = M_b^{\rightarrow} - M_s^{\rightarrow}$; $M_a^{\rightarrow}$ est la réunion dénombrable de graphes de **F** t.a. accessibles.

– $M_{s,r}^{\rightarrow} = M_s^{\rightarrow} \cap \{X_{t-} \in F\}$; $t \in M_{s,r}^{\rightarrow} \Leftrightarrow X_t \in F^c$, $X_{t-} \in F$

– $M_{s,i}^{\rightarrow} = M_s^{\rightarrow} \cap \{X_{t-} \notin F\}$

Rappelons enfin le théorème de balayage que l'on peut trouver dans [4]:

R.6.: Il existe une mesure σ finie de transition de $(E, \mathscr{B}^*(E))$ vers $(\Omega, \mathbf{F}_\infty^0)$, $\hat{E}_X$, et une fonctionnelle additive continue, J, de support contenu dans F, tels que pour toute variable aléatoire positive Φ, telle que $\Phi([\delta]) = 0$, la projection duale optionnelle et prévisible de

$$\sum_{0 < g \leqq t} \Phi \circ \theta_g 1_{M_\pi^{\rightarrow}}(g) \quad \text{soit égale à} \quad \int_0^t dJ_s \hat{E}_{X_s}(\Phi).$$

Nous commencerons par étudier l'ensemble des sauts de $\hat{X}$, c'est-à-dire $\{(t,\lambda,\omega) | \hat{X}_{t-}(\lambda,\omega)$ existe et $X_{t-}(\lambda,\omega) \neq \hat{X}_t(\lambda,\omega)\}$.
Soient $\hat{\omega} = (\lambda, \omega)$, $t \in \mathbb{R}_+$ fixés et (s_n) une suite de $\mathbb{R}_+$ croissant vers t.

– Si $\lambda \geqq t$ $\hat{X}_{s_n}(\lambda,\omega) = (\lambda - s_n, X_0(\omega))$ et $\hat{X}_{t-}(\lambda,\omega) = (\lambda - t, X_0(\omega))$

– Si $t > \lambda$ alors $s_n \geqq \lambda$ à partir d'un certain rang et

$$\hat{X}_{s_n}(\lambda,\omega) = (C_{j_{s_n - \lambda}}(\omega) - (s_n - \lambda), X_{j_{s_n - \lambda}}(\omega)).$$

On distingue alors deux cas:

a) – La suite $j_{s_n - \lambda}(\omega)$ est strictement croissante, autrement dit

$$(t - \lambda, \omega) \in \bar{G} \quad \text{et} \quad \lim_{n \to \infty} j_{s_n - \lambda}(\omega) = i_{t-\lambda}(\omega),$$

alors

$$\hat{X}_{t-}(\lambda,\omega) = (B_{i_{t-\lambda}}(\omega) - (t - \lambda), X^*_{i_{t-\lambda}-}(\omega)) = (0, X^*_{i_{t-\lambda}-}(\omega)).$$

b) – La suite $j_{s_n - \lambda}(\omega)$ est stationnaire;

$$(t - \lambda, \omega) \in \bar{G}^c \quad \text{et} \quad \hat{X}_{t-}(\lambda,\omega) = (C_{i_{t-\lambda}}(\omega) - (t - \lambda), X_{i_{t-\lambda}}(\omega)).$$

En résumé:

$$\begin{aligned} \hat{X}_{t-}(\lambda,\omega) = {} & (\lambda - t, X_0(\omega))\, 1_{\{\lambda \geqq t\}} + (0, X_{i_{t-\lambda}-}(\omega))\, 1_G(t - \lambda, \omega) \\ & + (C_{i_{t-\lambda}}(\omega) - (t - \lambda), X_{i_{t-\lambda}}(\omega))\, 1_{G^c}(t - \lambda, \omega)\, 1_{\mathbb{R}_+^*}(t - \lambda). \end{aligned}$$

On peut maintenant décrire les sauts de $\hat{X}$ au dessus de ses points de non branchement, c'est-à-dire les sauts de $\hat{X}$ appartenant à l'ensemble $\hat{\mathscr{P}}$ décrit ci-dessous

$$\hat{\mathscr{P}} = \{(s,\lambda,\omega),\ \hat{X}_{s-}(\lambda,\omega) \neq \hat{X}_s(\lambda,\omega),\ \hat{X}_{s-}(\lambda,\omega) \notin \bar{B},\ \hat{X}_s(\lambda,\omega) \notin \bar{B}\}.$$

Proposition 10. *Soit $\hat{T}$ un $\hat{\mathbf{F}}$ t a tel que $[[\hat{T}]] \subset \hat{\mathscr{G}}$. Alors: $\hat{T}(\lambda, .) > \lambda$ et $\hat{T}_\lambda = \hat{T}(\lambda) - \lambda$ a son graphe inclus dans $\bar{G}$.*

Démonstration. Considérons d'abord $\mathscr{S}_1 = \{(\omega, \lambda, s) | (\omega, \lambda, s) \in \hat{\mathscr{G}}, \lambda \geqq s\}$.

Il résulte de l'étude ci-dessus que:

$$\mathscr{S}_1 = \{(s, \lambda, \omega), s = \lambda, (0, X_0) \neq (\bar{R}_0, X_{j_0}), X_0 \in F \cap B^c\}.$$

Si $\hat{T}$ est un $\hat{\mathbf{F}}$ *t a* tel que $[[\hat{T}]] \subset \mathscr{S}_1$ alors $X_{\hat{T}^-}(\lambda, \omega) = (0, X_0(\omega))$, et on a pour tout $\omega \in \Omega$:

$$P^{X_0(\omega)}[j_0 = 0, X_{j_0} = X_0(\omega)] = P^{X_0(\omega)}[\hat{X}_{\hat{T}}(\lambda, .) = \hat{X}_{\hat{T}^-}(\lambda, \omega)]$$

D'autre part puisque $X_0(\omega) \in F \cap B^c$, on a:

$$P^{X_0(\omega)}[j_0 = 0, X_{j_0} = X_0(\omega)] = 1$$

et aussi d'après la propriété de régénération en 0

$$P^{X_0(\omega)}[\hat{X}_{\hat{T}}(\lambda, .) = \hat{X}_{\hat{T}}(\lambda, \omega)] = P^x[\hat{X}_{\hat{T}}(\lambda, \omega) = \hat{X}_{\hat{T}^-}(\lambda, \omega)] = 0 \qquad P^x \text{ ps.}$$

Par suite $\mathscr{S}_1$ ne contient aucun graphe de temps d'arrêt et la première assertion en découle.

Soit maintenant:

$$\mathscr{S}_2 = \{(s, \lambda, \omega) \in \hat{\mathscr{G}}, s > \lambda, (s - \lambda, \omega) \notin \bar{G}\}.$$

Toujours d'après l'étude précédente, on a aussi:

$$\mathscr{S}_2 = \{(s, \lambda, \omega) | (C_{i_{s-\lambda}}(\omega) - (s - \lambda), X_{i_{s-\lambda}}(\omega)) \neq (C_{j_{s-\lambda}}(\omega) - (s - \lambda), X_{j_{s-\lambda}}(\omega)) (s - \lambda, \omega) \in \bar{G}^c, (C_{i_{s-\lambda}}(\omega) - (s - \lambda), X_{i_{s-\lambda}}(\omega)) \notin \bar{B}\}.$$

Or nous avons vu que P^x ps $\bar{G}^c = \bigcup_n]]B_{T_n}, C_{T_n}]]$ où (T_n) est une suite de t.a qui épuise les sauts de C, et si

$$s - \lambda \in]B_{T_n}(\omega), C_{T_n}(\omega)] \qquad i_{s-\lambda}(\omega) = T_n(\omega).$$

De plus si:

$$s - \lambda \in]B_{T_n}(\omega), C_{T_n}(\omega)[, \qquad j_{s-\lambda}(\omega) = i_{s-\lambda}(\omega) = T_n(\omega)$$

on a donc:

$$\mathscr{S}_2 = \bigcup_{n \in N} \{(s, \lambda, \omega); s - \lambda = C_{T_n}(\omega), (C_{jT_n}(\omega) - T_n(\omega), X_{jC_{T_n}}(\omega)) \neq (C_{iC_{T_n}}(\omega) - C_{T_n}(\omega), X_{iC_{T_n}}(\omega)) \text{ et } (C_{iC_{T_n}}(\omega) - C_{T_n}(\omega), X_{iC_{T_n}}(\omega)) \in \bar{B}^c\}$$

or puisque $i_{C_{T_n}} = T_n$ et $C_{iC_{T_n}} - C_{T_n} = 0$ on a finalement que

$$\mathscr{S}_2 = \bigcup_{n \in N} \{(s, \lambda, \omega), s - \lambda = C_{T_n}(\omega), D_{T_n}(\omega) > T_n(\omega), X_{T_n}(\omega) \in F\}$$

or

$$P^x\{X_{T_n} \in F, D_{T_n} > T_n\} = 0 \qquad \forall x \in E \qquad \text{et} \quad \mathscr{S}_2 = \emptyset \; P^x \text{ ps.}$$

Notons que les hypothèses H'_2 et H'_3 n'interviennent pas dans cette démonstration.

Dans le corollaire 6 nous avons donné une première caractérisation des $\hat{F}$ temps d'arrêts totalement inaccessibles. La décomposition de l'ensemble aléatoire $M_\pi^{\rightarrow}$ que nous avons précédemment rappelée d'affiner cette caractérisation.

Proposition 11. *Soit $\overline{T}$ un $\overline{\mathbf{F}}$ temps d'arrêt. $\overline{T}$ est totalement inaccessible si et seulement si, il existe une variable aléatoire $\mathbf{F}_\infty$ measurable positive S dont le graphe est inclus dans l'ensemble:*

$$(\mathscr{S} \cup M_\pi^{\rightarrow}) \wedge G \quad \text{et telle que } \overline{T} = B_s.$$

Demonstration. Définissons

$$\overline{T}_1 = \begin{cases} \overline{T} & \text{si } i_T = j_T \text{ et } D_{i_T} = j_T \\ +\infty & \text{sinon} \end{cases}$$

$$\overline{T}_2 = \begin{cases} \overline{T} & \text{si } i_T = j_T \text{ et } D_{i_T} < j_T \\ +\infty & \text{sinon} \end{cases}$$

$$\overline{T}_3 = \begin{cases} \overline{T} & \text{si } i_T < j_T \\ +\infty & \text{sinon} \end{cases}.$$

$\overline{T}_1, \overline{T}_2$ et $\overline{T}_3$ sont manifestement des $\overline{\mathbf{F}}$ temps d'arrêts et $\overline{T}$ est totalement inaccessible si et seulement si $\overline{T}_1$, $\overline{T}_2$ et $\overline{T}_3$ sont totalement inaccessibles. Il suffit donc pour établir la condition nécessaire de la proposition, de le faire pour des temps d'arrêts tels que $\overline{T}_1$, $\overline{T}_2$ et $\overline{T}_3$.

Supposons d'abord que $i_T = j_T$, $D_{i_T} = j_T$ sur $\{\overline{T} < \infty\}$ alors d'après le rappel R.2., le graphe du $\mathbf{F}$ temps d'arrêt i_T est inclus dans $\mathscr{S} \cap G \cap \{(t, \omega) | t = D_t(\omega)\}$.

De même si $i_T = j_T$, $D_{iT} > j_T$ sur $\{\overline{T} < \infty\}$, le graphe du ta i_T est inclus dans $\mathscr{S} \cap G \cap M^{\rightarrow} = \mathscr{S} \cap G \cap \{t < D_t\}$.

Si maintenant $i_T < j_T$ sur $\{\overline{T} < \infty\}$, on ne peut plus affirmer que i_T soit un $\mathbf{F}$ temps d'arrêt, mais par contre on sait, toujours d'après le rappel précité, que $\overline{T} = \ell_{i_T}$ et que le graphe de i_T est disjoint de tout graphe de t.a. prévisible. De plus on a $i_T < j_T = j_{B_{i_T}} \leqq j_{C_{i_T}} = D_{i_T}$.

Il en résulte que le graphe de i_T est inclus dans $(M^{\rightarrow} - M_a^{\rightarrow}) \cap G$.

On a donc prouvé que si $\overline{T}$ est $\overline{\mathbf{F}}$ totslement inaccessible la variable aléatoire i_T a son graphe inclus dans $\mathscr{S} \cap G \cup (M^{\rightarrow} - M_a^{\rightarrow}) \cap G$, et vérifie $\overline{T} = B_{i_T}$. La réciproque de cette assertion découle immédiatement du rappel précité.

Il ne nous reste qu'à montrer que l'ensemble aléatoire décrit dans l'énoncé de la proposition est égal à $\mathscr{S} \cap G \cup (M^{\rightarrow} - M_a^{\rightarrow}) \cap G$.

Pour cela commençons par établir la relation:

$$D_s = j \circ \theta_s + s \quad \text{si } s \in \overline{M}.$$

On a $j \circ \theta_s = \text{Inf}\{u | C_u \circ \theta_s > 0\}$, or si $s \in \overline{M}$,

$$C_u \circ \theta_s = C_{u+s} - C_s \quad \text{et } j \circ \theta_s = \text{Inf}\{u | C_{u+s} > C_s\} = j\, C_s - s = D_s - s.$$

Il s'ensuit que si T est $\mathbf{F}$ t.a. dont le graphe est contenu dans $\overline{M}$:

$$X_T \in F \; P^x \text{ ps} \Leftrightarrow D_T = TP^x \text{ ps}.$$

En effet d'après la propriété de Markov forte, on a:

$$E^x(E^{X_T}(j_0=0))=E^x(j_0\circ\theta_T=0)=E^x(D_T=T).$$

Par suite on a:

$-)$ $\mathcal{S}\cap G\cap\{t=D_t\}=\mathcal{S}\wedge\rho_F\wedge G$

$-)$ $\mathcal{S}\cap G\cap\{t<D_t\}=\mathcal{S}\wedge\rho_F^c\wedge G$

$-)$ $(M^{\rightarrow}-M_a^{\rightarrow})\wedge G=M_\pi^{\rightarrow}\wedge G\cup\mathcal{S}\wedge\rho_F^c\wedge G$

ce qui prouve la proposition.

Posons

$$\bar{\mathcal{P}}^i=\{(t,\omega)|(i_t(\omega),\omega)\in(\mathcal{S}\cup M_\pi^{\rightarrow})\wedge G\}$$
$$\hat{\mathcal{P}}^i=\{(t,\lambda,\omega)|(t-\lambda,\omega)\in\bar{\mathcal{P}}^i\}.$$

Le corollaire 6 et la proposition 11 peuvent alors se formuler de la façon suivante:

Corollaire 12. *Soit $\hat{T}$ (resp. $\bar{T}$) un $\hat{\mathbf{F}}$ (resp. $\bar{\mathbf{F}}$) temps d'arrêt. $\hat{T}$ (resp. $\bar{T}$) est totalement inaccessible si et seulement si son graphe est inclus dans $\hat{\mathcal{P}}^i$ (resp. $\bar{\mathcal{P}}^i$).*

On peut aussi exprimer une relation entre les ensembles $\hat{\mathcal{P}}^i$ et $\bar{\mathcal{P}}^i$, et les temps de sauts de processus, ce sera l'objet de la proposition suivante.

Proposition 13. 1) *L'ensemble $\bar{\mathcal{P}}^i$ est inclus dans l'ensemble des sauts du processus (X_{j_t},j_t).*

2) *Si $(t,\lambda,\omega)\in\hat{\mathcal{P}}^i$, la limite à gauche $\hat{X}_{t-}(\lambda,\omega)$ existe et*

$$\hat{P}^{(\lambda,x)}\{(t,\lambda,\omega)|(t,\lambda,\omega)\in\hat{\mathcal{P}}^i,\hat{X}_{t-}(\lambda,\omega)\in\bar{B}\}=0$$

pour tout $(\lambda,x)\in.\,\mathbb{R}_+\times E$.

Démonstration. L'assertion 1) résulte de la décomposition de $\bar{\mathcal{P}}^i$ obtenue dans la proposition 13: si $(t,\omega)\in\bar{\mathcal{P}}^i$ et $i_t(\omega)=j_t(\omega)$ alors $X_{j_t-}(\omega)=X_{i_t-}(\omega)\neq X_{i_t}(\omega)=X_{j_t}(\omega)$, et si $i_t(\omega)<j_t(\omega)$ le résultat est immédiat.

Soit maintenant $(t,\lambda,\omega)\in\hat{\mathcal{P}}^i$, $\bar{R}$. étant un processus cadlag il suffit de vérifier que le terme lié au processus X admet une limite $X_{i_{t-\lambda}-}(\omega)$ dans E. Or d'après la proposition 11: $i_{t-\lambda}\in\mathcal{S}\cup M_\pi^{\rightarrow}$ et on sait alors que $\hat{X}^*_{i_{t-\lambda}-}$ existe. Le seul point non immédiat est le fait que cette limite $\hat{X}_-$ ne soit pas un point de branchement. D'après la description de $\bar{B}$, il nous suffit en fait de montrer que $X_{i_{(t-\lambda)}-}(\omega)$ n'appartient pas à $F\cap B$, P^x ps. D'après la proposition 11, cela revient à établir que:

i) $$1_{\{X_{g^-}\notin F\}}1_{\{X_g\in F\}}1_G(g)\,1_{\{D_g>g\}}=0 \quad P^x\text{ ps}$$

et que

ii) $$1_{\{X_{g^-}\notin F\}}1_{\{X_g\in F^c\}}1_{\{X_g\neq X_{g^-},X_{g^-}\in B\}}1_G(g)=0 \quad P^x\text{ ps.}$$

Prouvons la deuxième relation, à cet effet considérons une fonction h

borélienne strictement positive telle que:

$$E^0\Big(\sum_{0<s<\infty} h(X_{s-}, X_s)\Big) < \infty$$

(l'existence d'une telle h est établie dans [1]).

D'après le rappel R.4., on peut écrire que:

$$E^0\Big(\sum_{0<g<\infty} 1_G(g)\, 1_{(X_{g^-}\notin F)}\, 1_{(X_g\in F)} h(X_{g^-}, X_g)\Big)$$
$$= E^0 \int_0^\infty dL_s\, 1_G(s) \int_E N(X_{s^-}, dy)\, 1_{(X_{s^-}\notin F)}\, 1_{(y\in F)} h(X_{s^-}, y).$$

Or $\{X_{s-}\notin F\}\wedge G$ ne diffère de $\{s=D_s\}\cap\{X_s\notin F\}\wedge G$ que sur une réunion dénombrable de graphes de temps d'arrêts. De plus nous avons remarqué que $\rho_F = \{X_s\in F\}$ est la projection optionnelle de $\{s=D_s\}$ ces deux ensembles ne diffèrent donc que sur un ensemble à coupe dénombrable. Par suite $1_G(s)$. $1_{(X_{S^-}\notin F)}$ est sauf sur un ensemble à coupes dénombrables qui n'est pas chargé par la fonctionnelle continue L, donc

$$E^0\Big(\sum_{0<g<\infty} 1_G(g)\, 1_{(X_g\notin F)}\, 1_{(X_g\in F)} h(X_{g^-}, X_g)\Big) = 0$$

et ceci prouve la relation ii). La démonstration de i) s'effectue de façon analogue en utilisant le theorème de balayage (R.6).

Remarques: 1) Si $(s,\lambda,\omega)\in\mathscr{S}^i$, on n'a pas nécessairement:

$$\hat{X}_{s-}(\lambda,\omega)\neq\hat{X}_s(\lambda,\omega), \qquad \hat{X}_{s-}(\lambda,\omega)\notin\overline{B}.$$

Si $i_{s-\lambda}\notin M_\pi^{\rightarrow}$ cette relation est vérifiée mais par contre si $i_{s-\lambda}\in M_\pi^{\rightarrow}$ ce n'est en général pas vrai, il suffit que l'excursion en dehors de M soit une boucle et que l'extrémité droite ne soit pas un saut de C. Ceci n'est pas contradictoire avec le fait que le processus $\hat{X}$ puisse être droit car on n'a pas considéré ici les tribus canoniques de ce processus.

2) Par contre un phénomène opposé peut avoir lieu: l'ensemble

$$\{(t,\lambda,\omega)|\hat{X}_{t-}\neq\hat{X}_t,\ \hat{X}_{t-}\in\overline{B}^c\}\wedge\hat{\mathscr{P}}^{ic},$$

des sauts de $\hat{X}$ (pour la topologie $\hat{\tau}$) au dessus de ses points de non branchement, qui ne sont pas totalement inaccessibles peut n'être pas vide et même contenir un temps d'arrêt prévisible. Ceci n'est pas non plus contradictoire avec le fait que $\hat{X}$ puisse être un processus droit. Dans ce cas cela tient moins au fait que l'on n'ait pas considéré les tribus canoniques (on pourrait préciser les rapports entre les $\hat{\mathbf{F}}$ t.a. prévisibles et les temps d'arrêts prévisibles du processus à l'aide par exemple de [5]) mais plutôt comme l'a fait remarquer J.M. Bismut parce que la topologie de $\overline{E}$ n'est pas nécessairement la même que celle du compactifié de Ray de $\hat{X}$ et que la compactification peut gommer des sauts de la topologie initiale.

Un exemple de cette situation est donné dans [15].

IV. Systeme de Levy de $\hat{\mathbf{X}}$.

On désigne toujours par E l'espace $\mathbb{R}_+ \times E$ et par E l'espace $\hat{E} - \hat{B}$.

Soit f une fonction positive définie sur $\bar{E} \times \bar{E}$, borélienne nulle sur $(\tilde{E} \times \tilde{E})^c$ et sur la diagonale. On se propose de calculer la projection $\hat{\mathbf{F}}$ duale prévisible de la fonctionnelle:

$$\hat{K}_t = \sum_{0 < s \leqq t} f(\hat{X}_{s-}, \hat{X}_s) 1_{\hat{\mathcal{F}}^i}(s).$$

Pour établir ce résultat nous procéderons en plusieurs étapes.

−) Dans la première nous montrons que l'on peut ramener cette étude à celle de la projection duale $\bar{\mathbf{F}}$ prévisible d'un processus $\bar{K}$ croissant dont le support est inclus dans $\bar{G}$.

−) On sait alors grâce au rappel R.3. exprimer une telle projection en fonction de la projection duale $\mathbf{F}$ prévisible d'un processus croissant K.

−) On montre ensuite que l'on peut décomposer cette fonctionnelle en une somme de processus croissants dont on sait calculer explicitement les projections $\mathbf{F}$ duales prévisibles au moyen de résultats sur les systèmes de Levy de processus droits et sur le balayage (Rappels R.4.–R.6.).

Lemme 15. *Désignons par $\bar{K}_t$ le processus croissant:*

$$\sum_{0 < u \leqq t} f(0, X_{i_u -}, C_{j_u} - u, X_{j_u})(.) 1_{\mathcal{F}^i}(u, .)$$

et par $\bar{K}^p_t$ sa projection duale $\bar{\mathbf{F}}$ prévisible. Le processus croissant $\hat{K}^p_t$ défini par:

$$\hat{K}^p_t(\lambda, \omega) = \bar{K}^p_{t-\lambda}(\omega) 1_{\{t > \lambda\}}.$$

$((\lambda, \omega) \in \hat{\Omega})$ est alors pour toute loi $\hat{P}^{(\lambda, x)} = \varepsilon_\lambda \otimes P^x$, la projection duale $\hat{\mathbf{F}}$ prévisible de $\hat{K}_t$.

Démonstration: Soit (λ, x) fixé et $\hat{Z}$ un processus $\hat{\mathbf{F}}$ prévisible. On a alors:

$$\hat{E}^{(\lambda, x)} \int_0^\infty \hat{Z}_t \, d\hat{K}_t = E^x \int_0^\infty \hat{Z}_t(\lambda, .) \, d\hat{K}_t(\lambda, .)$$

puis

$$E^x \int_0^\infty \hat{Z}_t(\lambda, .) \, d\hat{K}_t(\lambda, .) = E^x \int_0^\lambda \hat{Z}_t(\lambda, .) \, d\hat{K}_t(\lambda, .) + E^x \int_0^\infty \hat{Z}_{t+\lambda}(\lambda, .) \, d\hat{K}_{t+\lambda}(\lambda, .)$$

or d'après la proposition 10 $K_t(\lambda, \omega) = 0$ si $t \leqq \lambda$, par suite

$$\hat{E}^{(\lambda, x)} \int_0^\infty \hat{Z}_t \, d\hat{K}_t = E^x \int_0^\infty \hat{Z}_{t+\lambda}(\lambda, \omega) \, d\hat{K}_{t+\lambda}(\lambda, \omega).$$

Il résulte alors de la définition de $\bar{\mathcal{F}}^i$ et $\hat{\mathcal{F}}^i$ que:

$$\hat{K}_{t+\lambda}(\lambda, \omega) = \bar{K}_t(\omega) = \sum_{0 < u \leqq t} f(0, X_{i_u -}, C_{j_u} - u, X_{j_u})(\omega) 1_{\mathcal{F}^i}(u, \omega)$$

Posons alors $\bar{Z}_t(\omega)=\hat{Z}_{t+\lambda}(\lambda,\omega)$, c'est d'après la proposition 5.e. un processus $\bar{\mathbf{F}}$ prévisible.

On a donc:

$$\hat{E}^{(\lambda,x)}\int_0^\infty \hat{Z}_t d\hat{K}_t = E^x\int_0^\infty \bar{Z}_t d\bar{K}_t$$

et

$$E^x\int_0^\infty \bar{Z}_t d\bar{K}_t = E_x\int_0^\infty \bar{Z}_t d\bar{K}_t^p$$

par définition de la projection duale prévisible.

Un calcul analogue prouve alors que $\hat{K}_t^p$ étant défini comme dans l'énoncé du lemme on a aussi:

$$E^x\int_0^\infty \bar{Z}_t d\bar{K}_t^p = \hat{E}^{(\lambda,x)}\int_0^\infty \hat{Z}_t d\hat{K}_t^p$$

et puisque $\hat{K}^p$ est un processus croissant prévisible le lemme est établi.

Nous nous sommes donc ramenés à l'étude de la projection duale $\bar{\mathbf{F}}$ prévisible de $\bar{K}_t$.

Notons que le support de $\bar{K}_t$ est inclus dans $\bar{G}$.

Le théorème 9 de [7] nous donne un moyen explicite de calculer sa projection duale prévisible. Il suffit d'introduire le processus $K_t=\bar{K}_{C_t}$ et on alors $\bar{K}_t^p=(K^p)_{i_t}=(K^p)_{j_t}$.

Nous allons donc calculer K^p, à cette effet nous commencerons par établir le lemme suivant:

Lemme 16. *Posons*

$$K_t^1=\sum_{0<g\leq t} f(0,X_g,\Delta C_g,X_g)1_{\mathscr{I}}(g)\,1_G(g)$$

$$K_t^2=\sum_{0<g\leq t} f(0,X_g,C_{D_g}-C_g,X_{D_g})1_{M_{\bar{T}}}g$$

$$K_t^3=\sum_{0<g\leq t} f(0,X_{g-},C_{D_g}-C_g,X_{D_g})1_{M_{\bar{T},r}^-}(g)1_{G\wedge\Gamma^c}(g)$$

où Γ désigne l'ensemble aléatoire des sauts de C, c'est-à-dire

$$\Gamma=\{(\omega,s);\Delta C_s(\omega)\neq 0\}$$

Alors $K_t=K_t^1+K_t^2+K_t^3$.

Démonstration. On a tout d'abord:

$$\begin{aligned}K_t&=\sum_{0<u\leq C_t} f(0,X_{i_u-},C_{j_u}-u,X_{j_u})1_{\mathscr{I}}(u)\\&=\sum_{0<u\leq C_t} f(0,X_{i_u-},C_{j_u}-u,X_{j_u})1_{(i_u=j_u)}1_{\mathscr{I}}(u)\\&\quad+\sum_{0<u\leq C_t} f(0,X_{i_u-},C_{j_u}-u,X_{j_u})1_{(i_u<j_u)}1_{\mathscr{I}}(u).\end{aligned}$$

Commençons par remarquer que si $u=B_{i_u}$ et $i_u<j_u$ alors $i_u\notin\Gamma$ et donc que $D_{i_u}=j_{C_{i_u}}=j_{B_{i_u}}=j_u$.

D'autre part si $u\in\bar{G}$, $\ell_{i_u}=i_u$

Il résulte alors de la relation $u\leqq C_t \Leftrightarrow i_u\leqq t$ et de la proposition (11) que:

$$\sum_{0<u\leqq C_t} f(0,X_{i_u-},C_{j_u}-u,X_{j_u})1_{(i_u<j_u)}1_{\mathscr{P}^i}(u)$$
$$=\sum_{0<g\leqq t} f(0,X_{g-},C_{D_g}-C_g,X_{D_g})1_G(g)1_{(M^{\rightarrow}-M_a^{\rightarrow})}(g)1_{\Gamma^c}(g).$$

Or $G\cap(M^{\rightarrow}-M_a^{\rightarrow})=G\cap(M_\pi^{\rightarrow}\cup M_{s,r}^{\rightarrow})$ et si $g\in M_\pi^{\rightarrow}$ on sait que $X_g=X_{g-}$.

Par ailleurs, $M_\pi^{\rightarrow}\cap\Gamma^c=M_\pi^+$ puisque aucun graphe de t.a. ne passa dans $M_\pi^{\rightarrow}$ (ps).

Par suite:

$$\sum_{0<u\leqq C_t} f(0,X_{i_u-},C_{j_u}-u,X_{j_u})1_{(i_u<j_u)}1_{\mathscr{P}^i}(u)=K_t^2+K_t^3 \text{ ps.}$$

Montrons enfin que:

$$\sum_{0<u\leqq C_t} f(0,X_{i_u-},C_{j_u}-u,X_{j_u})1_{(i_u=j_u)}1_{\mathscr{P}^i}(u)$$
$$=\sum_{0<g\leqq t} f(0,X_{g-},\Delta C_g,X_g)1_{\mathscr{S}}(g)\,1_G(g)$$

Il résulte de la proposition 11 que si $u\in\overline{\mathscr{P}^i}$ et $i_u=j_u=g$ on a $g\in\mathscr{S}\wedge G$ et

$$f(0,X_{i_u-},C_{j_u}-u,X_{j_u})=f(0,X_{g-},C_g-B_g,X_g).$$

Réciproquement si $g\in\mathscr{S}\wedge G$, on a toujours d'après la proposition précitée $u=B_g\in\overline{\mathscr{P}^i}$ et:

i) si $Dg=g: g=i_{B_g}\leqq j_{B_g}\leqq j_{C_g}=g$ soit $i_{B_g}=j_{B_g}$.

ii) si $g<D_g$ et $i_u<j_u$ alors nécessairement $g\notin\Gamma$ et par suite si $g<D_g$ et $X_g\notin M_\pi^{\rightarrow}$ alors $X_g\notin\hat{F}$ et $(\Delta C_g,X_g)=(0,X_g)\in\hat{B}$ et donc $f(0,X_{g-},\Delta C_g,X_g)=0$.

L'égalité requise résulte alors de la relation $\{u\leqq C_t\}\Leftrightarrow\{i_u\leqq t\}$.

Notons enfin une conséquence immédiate du rappel R.5.:

Lemme 17. *Il existe une fonction positive α sur $E\times E$, $\mathscr{E}'\otimes\mathscr{E}$ mesurable telle que:*

$$\sum_{0<s\leqq t}\Delta C_s 1_{\mathscr{S}}(s)=\sum_{0<s\leqq t}\alpha(X_{s-},X_s)1_{\mathscr{S}}(s).$$

Nous pouvons maintenant exprimer les projections duales prévisibles des processus K_t^i $(i=1,2,3)$.

Calcul de K_t^{1p}: Posons $h(a,b)=f(0,a,\alpha(a,b),b)$. Alors d'après le lemme 17

$$K_t^1=\sum_{0<g\leqq t} f(0,X_{g-},\Delta C_g,X_g)1_{\mathscr{S}\wedge G}(g)$$
$$=\sum_{0<g\leqq t} h(X_{g-},X_g)1_{\mathscr{S}\wedge G}(g)$$

et G étant un ensemble prévisible on à d'après R.5.:

$$K_t^{1p}=\int_0^t 1_G(s)\,dL_s\int N(X_{s-},dy)\,h(X_{s-},y)$$

soit encore

$$K_t^{1p}=\int_0^t 1_G(s)dL_s\int N(X_{s-},dy)\,f(0,X_{s-},\alpha(X_{s-},y),y).$$

Calcul de K_t^{2p}: Il résulte de la relation $D_s=j_0\circ\theta_s+s$ (démonstration de la proposition (11) que:

$$X_{D_g}=X_{j_0}\circ\theta_g \quad \text{et que} \quad C_{D_g}-C_g=C_{j_0}(\theta_g)$$

Posons

$$H(\omega)=f(0,X_0(\omega),C_{j_0}(\omega),X_{j_0}(\omega))$$

alors

$$K_t^2=\sum_{0<g\leq t} H\circ\theta_g 1_{M_{\pi}^{\rightarrow}}(g)$$

est une fonctionnelle additive homogène à laquelle on applique le théorème de balayage R.6. On obtient:

$$K_t^{2p}=K_t^{20}=\int_0^t dJ_s 1_G(s)\hat{E}_{X_s}(f,X_0,C_{j_0},X_{j_0})).$$

Calcul de K_t^{3p}: Notons d'abord que:

$$K_t^3=\sum_{0<g\leq t} f(0,X_{g-},C_{D_g}-C_g,X_{D_g})1_{M_{s,r}^{\rightarrow}\wedge G}(g)1_{\Gamma^c}(g)=K_t^4-K_t^5$$

où

$$K_t^4=\sum_{0<g\leq t} f(0,X_{g-},C_{D_g}-C_g,X_{Dg})1_{M_{s,r}^{\rightarrow}\wedge G}(g)$$

et

$$K_t^5=\sum_{0<g\leq t} f(0,X_{g^-}-C_g,X_{D_g})\,1_{M_{s,r}^{\rightarrow}AG}(g)\,1_{\Gamma}(g)$$

Calculons d'abord K_t^{4p}. Nous commencerons par supposer que $f=f_1\times f_2$, $f_1,f_2\in b\hat{E}$.

$$K_t^4=\sum_{0<g\leq t} f_1(0,X_{g-})f_2(C_{D_g}-C_g,X_{D_g})1_{M_{s,r}^{\rightarrow}\wedge G}(g)$$

Or

$$f_2(C_{D_g}-C_g,X_{Dg})=f_2(C_{j_0},X_{j_0})\circ\theta_g\ldots$$

On sait alors ([6]) que la projection duale prévisible du processus

$$\sum_{0<g\leq t} f_2(C_{j_0},X_{j_0})\circ\theta_g 1_{M_{s,r}^{\rightarrow}}(g)$$

est égale à

$$\int_0^t dL_s\int N(X_s,dy)\,E_y[f_2(C_{j_0},X_{j_0})]1_{\Gamma^c}(y).$$

Le processus $\sum_{0<g\leq t} f_1(0,X_{g-})1_G(g)$ étant prévisible, le processus K_t^{4p} vaut donc:

$$K_t^{4p}=\int_0^t 1_G(s)dL_s\int N(X_s,dy)E_y[f(0,X_{s-},C_{j_0},X_{j_0})]1_{\Gamma^c}(y).$$

Ce résultat subsiste pour tout f par classe monotone.

Quant à K_t^5 notons que:

$$1_{M_{s,r}^{\rightarrow}}(g)1_I(g)=1_{M_{s,r}^{\rightarrow}}(g)1_{\mathscr{S}}(g)1_{\{\Delta C_g\neq 0\}}=1_{M_{s,r}^{\rightarrow}}(g)1_{\{\Delta C_g 1_{\mathscr{S}}(g)\neq 0\}}$$

et puisque les processus $\sum_{0<g\leqq t}\Delta C_g 1_{\mathscr{S}}(g)$ et $\sum_{0<s\leqq t}\alpha(X_{s-},X_s)$ sont indistinguables, il en est de même pour les processus:

$$K_t^5 \quad \text{et} \quad \sum_{0<g\leqq t} f(0,X_{g-},C_{D_g}-C_g,X_{D_g})1_G(g)1_{M_{s,r}^{\rightarrow}}(g)1_{\{\alpha(X_{g-},X_g)\neq 0\}}$$

Mais $M_{s,r}^{\rightarrow}=\{(g,\omega)|g<D_g(\omega),\ X_{g-}\in F,\ X_g\notin F\}$, ce qui permet d'écrire que:

$$K_t^5=\sum_{0<g\leqq t} f(0,X_{g-},C_{D_g}-C_g,X_{D_g})\,1_G(g)1_{\{g<D_g,\,X_{g-}\in F,\,X_g\notin F\}}1_{\{\alpha(X_{g-},X_g)>0\}}$$

Posons

$$\ell(x,y)=E_y f(0,x,C_{j_0},X_{j_0})1_{\{j_0>0\}}$$

et

$$K_t^6=\sum_{0<g\leqq t}\ell(X_{g-},X_g)1_{\{X_{g-}\in F,\,X_g\notin F\}}1_G(g)1_{\{\alpha(X_{g-},X_g)\neq 0\}}.$$

K_t^6 est la projection duale optionnelle de K^5, par suite les projections duales prévisibles de K^5 et K^6 sontidentiques. Or la théorie des systèmes de Levy permet de calculer aisément K_t^{6p}:

$$K_t^{6p}=\int_0^t dL_s 1_G(s)1_{\{X_{s-}\in F\}}\int N(X_{s-},dy)\,\ell(X_{s-},y)1_{\{\alpha(X_{s-},y)\neq 0\}}1_{F^c}(y).$$

Résumons le long calcul qui précède:

Théorème 17. *Pour toute loi $\hat{P}^{\hat{\mu}}$, la projection duale $\hat{\mathbf{F}}$ prévisible du processus croissant $\hat{K}_t=\sum_{0<s\leqq t} f(\hat{X}_{s-},\hat{X}_s)1_{\hat{\mathscr{S}}^1}(s)$ est le processus continu $\hat{K}_t^{\hat{p}}$ défini par:*

$$\hat{K}_t^{\hat{p}}(\lambda,\omega)=(U)_{i_{t-\lambda}}(\omega)\,1_{\{t>\lambda\}}=U_{j_{t-\lambda}}(\omega)\,1_{\{t>\lambda\}}$$

où U est le processus croissant $\mathbf{F}$ adapté continu défini par:

$$U(\omega)=\int_0^t 1_G(s)\,dL_s(\omega)\int N(X_{s-}(\omega),dy)\,\{[f(0,X_{s-}(\omega),\alpha(X_{s-}(\omega),y),y)]$$
$$-1_{F^c}(y)\,E^y[f(0,X_{s-},C_{j_0},X_{j_0})(1_{\{\alpha(X_{s-}(\omega),X_0)\neq 0\}}]\}$$
$$+\int_0^t 1_G(s)\,dJ_s(\omega)\,\hat{E}_{X_s}(\omega)[f(0,X_0,C_{j_0},X_{j_0})]$$

où L et J sont des fonctionnelles continues, N est le noyau de Levy du processus X et où α est une fonction positive sur $E\times E-\Delta$ vérifiant

$$\sum_{0<s\leqq t}\Delta C_s 1_{\mathscr{S}}(s)=\sum_{0<s\leqq t}\alpha(X_{s-},X_s).$$

Corollaire 18. *Le processus $\hat{K}^{\hat{p}}$ défini précédemment est pour toute loi $\hat{P}^{\hat{\mu}}$ la*

projection duale **G** *prévisible du processus croissant:*

$$\hat{K}_t = \sum_{0<s\leq t} f(\hat{X}_{s^-}, \hat{X}_s)\, 1_{\mathcal{G}_1}(s).$$

(*Rappelons que* $\mathbf{G}_t$ *désigne la filtration canonique de* $\hat{X}$.)

Démonstration. Il suffit évidemment de montrer que $\hat{K}^{\hat{p}}$ est **G** adaptée. Pour cela nous reprendrons un argument qui figure dans [5], théorème 14.

Un calcul simple mais fastdieux permet d'établir que $\hat{K}^{\hat{p}}$ est une $\hat{\theta}$ fonctionnelle additive. Ceci permet d'écrire que:

$$\hat{E}^{\hat{\mu}}\left(\int_t^\infty e^{-s}\, d\hat{K}_s^{\hat{p}} | \hat{\mathbf{F}}_s\right) = e^{-t}\,\hat{E}^{\hat{\mu}}\left(\int_0^\infty e^{-s}\, d\hat{K}_s^{\hat{p}} \circ \hat{\theta}_t | \hat{\mathbf{F}}_s\right)$$

$$= e^{-t}\,\hat{E}^{\hat{\mu}}\left(\hat{E}^{\hat{X}_t} \int_0^\infty e^{-s}\, d\hat{K}_s^{\hat{p}}\right).$$

Posons $u(\lambda, x) = \hat{E}^{(\lambda, x)}\left(\int_0^\infty e^{-s}\, d\hat{K}_s^{\hat{p}}\right)$. La fonction u est universellement mesurable et le processus $u(\hat{X}_t)$ est **G** adapté.

D'autre part $u(\hat{X}_t)$ est un potentiel de la classe D d'après la relation précédente. La formule de Meyer sur les Laplaciens approchés (cf. par exemple [3]) montre que puisque $\hat{K}^{\hat{p}}$ est continue, pour toute loi μ, $\hat{P}^{\hat{\mu}}$ ps.

$$\int_0^t e^{-s}\, d\hat{K}_s^{\hat{p}} = \lim_{L^1(P^\mu)} \frac{1}{h}\left\{\int_0^t e^{-s}(u(\hat{X}_s) - e^{-h}\,\hat{E}^{\hat{X}_s}[u(\hat{X}_h)]\, ds.\right\}$$

Les processus qui interviennent dans le membre de droite sont $\mathbf{G}_t$ mesurables et leur limite $\int_0^t e^{-s}\, d\hat{K}_s^{\hat{p}}$, indépendante de μ est aussi $\mathbf{G}_t$ mesurable puisque les tribus $\mathbf{G}_t$ sont $\hat{P}^{\hat{\mu}}$ complètes pour toute loi $\hat{\mu}$. Il s'ensuit que $\hat{K}^{\hat{p}}$ est **G** adapté.

Remarque. En fait nous n'avons pas calculé le système de Levy au sens habituel du terme: dans le processus croissant $\hat{K}_t$ la sommation ne porte que sur les temps d'arrêts totalement inaccessibles des tribus $\hat{\mathbf{F}}_t$ plus «grosses» que les tribus canoniques $\mathbf{G}_t$.

Lorsque $\hat{X}$ est un processus droit, on peut établir (cf. [5]) que $\hat{K}^{\hat{p}}$ est bien la projection duale **G** prévisible du processus croissant $\sum_{0<s\leq t} f(\hat{X}_{s^-}, X_s)$ où la somme est effectuée sur tous les t.a. **G** totalement inaccessibles.

Par contre si l'on ne sait pas que $\hat{X}$ est droit, un tel résultat nécessite une étude des **G** temps d'arrêts totalement inaccessibles qui n'entre pas dans le cadre de ce travail.

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Consistency of Random Field Specifications

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Summary. In the random field approach to lattice gas models it has been shown that the one point conditional probabilities determine the finite set conditional probabilities under conditions of strict positivity and regularity. This paper considers the case when strict positivity does not obtain with families of conditional probabilities more general than the one-point conditional probabilities.

1. Introduction

Certain problems in the theory of random fields had their origins in lattice gas models of statistical mechanics. The basic structures are a countably infinite set of sites S and a finite set Y which describes the configuration of a single site, the overall configuration being described by a point of $X = Y^S$. In application to magnetic phenomena, S corresponds to the locations of atoms in a crystal and the point of Y gives the direction of the spin of a specific atom. In the physical model one prescribes an interaction potential among sites and determines the properties of probability measures on X which are consistent with the given potential (for details see Ruelle [8]).

Another approach to systems of this type employs conditional probabilities. Let $\mathcal{A}$ denote the set of finite subsets of S. For $\alpha \in \mathcal{A}$ we have the natural projection from X to Y^α. Let $\mathcal{F}_\alpha$ denote the σ-field of subsets of X which are inverse images of subsets of Y^α under this projection. Let $\mathcal{F}^\alpha$ denote the σ-field generated by $\{\mathcal{F}_\beta : \beta \in \mathcal{A}, \beta \cap \alpha = \emptyset\}$, and let $\mathcal{F}$ denote the σ-field generated by all $\mathcal{F}_\alpha$, $\alpha \in \mathcal{A}$. Corresponding to the probability measure μ on $(X, \mathcal{F})$ there is the conditional expectation operator $E(\cdot | \mathcal{F}^\alpha)$ for each $\alpha \in \mathcal{A}$. Since $(X, \mathcal{F})$ is a standard Borel space, $E(\cdot | \mathcal{F}^\alpha)$ can be expressed in terms of a regular conditional probability distribution $\mu(\omega = x \text{ on } \alpha \,|\, \omega = y \text{ on } S \setminus \alpha)$ for $\omega, x, y \in X$, $\alpha \in \mathcal{A}$ (see [6]).

In [3] Dobrushin considered the following problem. Given the real valued function $P_\alpha(x, y)$ for $\alpha \in \mathcal{A}$, $x, y \in X$, find all probability measures μ on $(X, \mathcal{F})$

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which satisfy

$$\mu(\omega = x \text{ on } \alpha \mid \omega = y \text{ on } S \setminus \alpha) = P_\alpha(x, y) \quad \mu\text{-a.e.} \tag{0}$$

This line of investigation has been extensively pursued (see Preston [7]).

In order for (0) to hold for some probability measure μ on $(X, \mathcal{F})$, the function $P_\alpha(x, y)$ must satisfy certain positivity, normalization and consistency conditions. It is convenient to define the equivalence relationship $x = y \bmod \alpha$ to mean that $x^j = y^j$ for $j \in S \setminus \alpha$, the superscript denoting component. For $\alpha \in \mathcal{A}$ the equivalence class mod α of a particular element $x \in X$ is a finite subset of X. We note that $P_\alpha(x, y)$ need only be defined for $x = y \bmod \alpha$. The positivity requirement is $P_\alpha(x, y) \geqq 0$. The normalization requirement is

$$\sum_x P_\alpha(x, y) = 1 \text{ each } y \in X,$$

where the sum is over those x equivalent to $y \bmod \alpha$. The consistency requirement is that for all $\alpha \in \mathcal{A}$, $y \in X$ and $\beta \subset \alpha$,

$$P_\beta(y, y) = P_\alpha(y, y) / \sum_x P_\alpha(x, y),$$

where the sum is over those x equivalent to $y \bmod \beta$. This means essentially that the conditional probabilities for $\beta \subset \alpha$ can be constructed from P_α in the usual way. All the above relations need hold only μ-almost everywhere.

The problem we consider in this paper is given $\{P_\beta(x, y)\}$ for all $\beta \in \mathcal{B}$ and $x = y \bmod \beta$, where $\mathcal{B}$ is a proper subset of $\mathcal{A}$, what are the consistency conditions on $\{P_\beta(x, y)\}$ and in how many ways can one construct a consistent $\{P_\alpha : \alpha \in \mathcal{A}\}$ which agrees with that given for $\alpha \in \mathcal{B}$. The most important case is when $\mathcal{B} = \{\{j\} : j \in S\}$. The collection $\{P_{\{j\}} : j \in S\}$ are called the one point conditional probabilities; the full collection $\{P_\alpha : \alpha \in \mathcal{A}\}$ are called the finite set conditional probabilities. Under assumption of strict positivity, together with regularity conditions, it has been shown that the one point conditional probabilities determine the finite set conditional probabilities and a corresponding potential can be constructed (see [2, 5, 10]).

In both the physical and mathematical contexts it is natural to consider models in which strict positivity fails. To deal with this case we introduce a set of allowed configurations $X_0 \subset X$ where strict positivity does obtain. For a given X_0 and a family $\{P_\alpha : \alpha \in \mathcal{B}\}$ we ask two questions: Are the given data consistent? Do they determine $\{P_\alpha : \alpha \in \mathcal{A}\}$ uniquely?

To motivate the techniques used below consider the case in which S is a finite set so that the space X is also finite. Let $\mathcal{B} = \{\{j\} : j \in S\}$. If (0) is satisfied and $P_{\{j\}}(x, y)$ is strictly positive, then $\mu(\{x\})/\mu(\{y\}) = P_{\{j\}}(x, y)/P_{\{j\}}(y, y)$ for $x = y \bmod j$. Now given $x, y \in X$ we can find a chain $x_0 = x, x_1, x_2, \ldots x_n = y$ where x_i and x_{i+1} differ at only one site. Now $\mu(\{x\})/\mu(\{y\})$ is the product of the rations $\mu(\{x_i\})/\mu(\{x_{i+1}\})$. Thus $\{P_{\{j\}}(x, y) : j \in S\}$ determines uniquely $\mu(\{x\})/\mu(\{y\})$ for $x, y \in X$. Normalization then determines μ uniquely. The consistency requirement is that any chain from x to y should yield the same value for $\mu(\{x\})/\mu(\{y\})$. To carry out these manipulations we need strict positivity. If however the set X_0 on which strict positivity obtains has the property that any two distinct elements

differ on at least two sites, then the one point conditional probabilities are trivial and the above method gives no information about μ.

When we consider the general case in the next section we shall recast the problem in algebraic form in order to avoid the difficulty of the relationships holding only almost everywhere. It is convenient to work in an abstract format involving partially ordered equivalence relations. In this context the extent to which $\{P_\alpha: \alpha \in \mathscr{B}\}$ determine $\{P_\alpha: \alpha \in \mathscr{A}\}$ and the consistency conditions have natural expressions in terms of concepts from homological algebra. This still leaves the problem of expressing the geometric constraints of a given model in terms of the algebraic criteria. For an important class of one dimensional systems we can give a reasonably satisfactory solution. Higher dimensional models pose difficult combinatorial problems.

2. Algebraic and Ratio Specifications

The basic structures for our algebraic approach are as follows. We have a set X_0 and a collection of partitions of X_0 into equivalence classes. The partitions are parametrized by the set $\mathscr{A}$, which itself is a collection of subsets of the set S. For each $\alpha \in \mathscr{A}$ and $x \in X_0$ we write $\{x\}_\alpha$ to denote the equivalence class of x corresponding to α; also we write $x = y \bmod \alpha$ for this equivalence. We require that $\{x\}_\alpha \subset \{x\}_\beta$ when $\alpha \subset \beta$ with $x \in X_0$, $\alpha, \beta \in \mathscr{A}$. Also we require that for each $\alpha, \beta \in \mathscr{A}$ there is a $\gamma \in \mathscr{A}$ with $\alpha \cup \beta \subset \gamma$. The above sets and equivalence classes will be denoted by the symbol $\mathscr{X}$.

2.1. *Definition.* An *algebraic specification* $P_\alpha(x, y)$ is a real valued function defined for all $\alpha \in \mathscr{A}$ and $x = y \bmod \alpha$, which satisfies

$$P_\alpha(x, y) > 0, \tag{1}$$

$$P_\alpha(x, y) = P_\alpha(x, x), \tag{2}$$

$$P_\alpha(x, y)\, P_\beta(y, x) = P_\beta(x, y)\, P_\alpha(y, x) \text{ when } \alpha \subset \beta \tag{3}$$

for all $\alpha, \beta \in \mathscr{A}$ and $x = y \bmod \alpha$.

This definition is motivated by conditional probabilities: (2) corresponds to measurability with respect to the appropriate σ-field and (3) corresponds to the consistency requirement (see Preston [7], Lemma 5.1).

2.2. *Definition.* Two algebraic specifications $P_\alpha(x, y)$ and $Q_\alpha(x, y)$ on $\mathscr{X}$ are said to be *equivalent* if

$$P_\alpha(x, y)\, Q_\alpha(y, x) = P_\alpha(y, x)\, Q_\alpha(x, y)$$

for all $\alpha \in \mathscr{A}$ and $x = y \bmod \alpha$.

In the conditional probability context one can often choose a particular element of each equivalence class by the requirement of normalization with respect to certain measures.

2.3. *Definition.* A *ratio specification* $F_\alpha(x, y)$ on $\mathscr{X}$ is a real valued function defined for all $\alpha \in \mathscr{A}$ and $x = y \bmod \alpha$ which satisfies

$$F_\alpha(x, y) > 0, \tag{4}$$

$$F_\alpha(x, y) F_\alpha(y, z) = F_\alpha(x, z), \tag{5}$$

$$F_\alpha(x, y) = F_\beta(x, y) \text{ when } \alpha \subset \beta \tag{6}$$

for all $\alpha, \beta \in \mathscr{A}$ and $x = y \bmod \alpha$, $y = z \bmod \alpha$.

2.4. **Lemma.** *An algebraic specification $P_\alpha(x, y)$ on $\mathscr{X}$ determines uniquely a ratio specification. A ratio specification $F_\alpha(x, y)$ on $\mathscr{X}$ determines an algebraic specification up to equivalence.*

Proof. Let $P_\alpha(x, y)$ be an algebraic specification on $\mathscr{X}$. Define $F_\alpha(x, y) = P_\alpha(x, y)/P_\alpha(y, x)$. Then (4) follows from (1) while (6) follows from (1) and (3). Also (5) follows from (1) and (2). To prove the second part we select an element $z(\alpha, x)$ from each equivalence class $\{x\}_\alpha$, *i.e.* $z(\alpha, x) = x \bmod \alpha$ and $x = y \bmod \alpha$ implies $z(\alpha, x) = z(\alpha, y)$. Now given the ratio specification $F_\alpha(x, y)$, define $P_\alpha(x, y) = F_\alpha(x, z(\alpha, x))$ for $x = y \bmod \alpha$. Then (1) and (2) follow from this definition and (4). We have $P_\alpha(x, y)/P_\alpha(y, x) = F_\alpha(x, y)$ by (5), so (6) then implies (3). It is not difficult to verify that different choices of $z(\alpha, x)$ give equivalent algebraic specifications and that equivalent algebraic specifications yield the same ratio specification.

Now we come to the basic problems. Assume that we are given $\mathscr{B} \subset \mathscr{A}$ and a function $P_\alpha(x, y)$ satisfying (1) and (2) for all $\alpha \in \mathscr{B}$ and $x = y \bmod \alpha$. Does there exist an extension of $P_\alpha(x, y)$ which is an algebraic specification on $\mathscr{X}$? Can there be nonequivalent extensions? We shall pose and answer these questions for ratio specifications because certain concepts from homological algebra arise naturally in this context.

For any $\mathscr{B} \subset \mathscr{A}$ let $C(\mathscr{B})$ denote the free abelian group generated by triples of the form (y, z, α) where $\alpha \in \mathscr{B}$ and $y = z \bmod \alpha$. Let $C(X_0)$ denote the free abelian group generated by the elements of X_0. We define the boundary homomorphism

$$\partial_{\mathscr{B}} \colon C(\mathscr{B}) \to C(X_0)$$

by

$$\partial_{\mathscr{B}} \left(\sum_{i=1}^{n} k_i(x_i, y_i, \alpha_i) \right) = \sum_{i=1}^{n} (k_i x_i - k_i y_i) \tag{7}$$

where the k_i's are integers. Elements in $\ker \partial_{\mathscr{B}}$ will be called *cycles*. Given a function $F_\alpha(x, y)$ defined for $\alpha \in \mathscr{B}$ and $x = y \bmod \alpha$ which satisfies (4), we can extend it to a homomorphism $F_{\mathscr{B}} \colon C(\mathscr{B}) \to R^+$, the positive real numbers under multiplication, by

$$F_{\mathscr{B}} \left(\sum_{i=1}^{n} k_i(x_i, y_i, \alpha_i) \right) = \prod_{i=1}^{n} (F_{\alpha_i}(x_i, y_i))^{k_i}. \tag{8}$$

2.5. **Lemma.** *$F_{\mathscr{A}}$ is trivial on $\ker \partial_{\mathscr{A}}$ if and only if $F_\alpha(x, y)$ satisfies (5) and (6).*

Proof. Suppose $F_{\mathscr{A}}$ is trivial on cycles. When $x = y \bmod \alpha$, $y = z \bmod \alpha$ and $\alpha \subset \beta$, we have the cycles $(x, y, \alpha) + (y, z, \alpha) - (x, z, \alpha)$ and $(x, y, \alpha) - (x, y, \beta)$. Apply $F_{\mathscr{A}}$ to these cycles to obtain (5) and (6). Conversely suppose (5) and (6) hold for $F_\alpha(x, y)$. Cycles of the form $\sum_{i=1}^{n} (x_i, x_{i+1}, \alpha_i)$ with $x_1 = x_{n+1}$ generate $\ker \partial_{\mathscr{A}}$. By our basic

assumptions on $\mathscr{X}$ there is a $\gamma \in \mathscr{A}$ containing $\alpha_1, \ldots, \alpha_n$. Using this γ and (5), it follows easily from (6) that $F_{\mathscr{A}}$ is trivial on cycles of this form. Since $F_{\mathscr{A}}$ is a homomorphism and these cycles generate $\ker \partial_{\mathscr{A}}$, we have the desired result.

2.6. **Theorem.** *Let $F_\alpha(x, y)$ be a real valued function defined for $\alpha \in \mathscr{B}$ and $x = y \bmod \alpha$ which satisfies* (4). *Then $F_\alpha(x, y)$ is the restriction to $\mathscr{B}$ of a ratio specification on $\mathscr{X}$ if and only if $F_{\mathscr{B}}$ is trivial on* $\ker \partial_{\mathscr{B}}$.

Proof. Let i: $C(\mathscr{B}) \to C(\mathscr{A})$ be the natural inclusion homomorphism. Then

$$\partial_{\mathscr{B}} = \partial_{\mathscr{A}} \circ i.$$

Thus there exists a monomorphism

$$k\colon C(\mathscr{B})/\ker \partial_{\mathscr{B}} \to C(\mathscr{A})/\ker \partial_{\mathscr{A}}$$

given by $k(a + \ker \partial_{\mathscr{B}}) = i(a) + \ker \partial_{\mathscr{A}}$. Suppose that $F_{\mathscr{B}}$ is trivial on $\ker \partial_{\mathscr{B}}$. Then we have the induced homomorphism

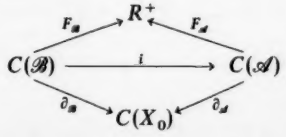
$$\tilde{F}_{\mathscr{B}}\colon C(\mathscr{B})/\ker \partial_{\mathscr{B}} \to \mathrm{R}^+.$$

Since the positive reals under multiplication is a divisible abelian group and k is a monomorphism, there exists a homomorphism

$$\tilde{F}_{\mathscr{A}}\colon C(\mathscr{A})/\ker \partial_{\mathscr{A}} \to R^+.$$

such that $\tilde{F}_{\mathscr{B}} = \tilde{F}_{\mathscr{A}} \circ k$ (see Theorem 21.1 of Fuchs [4]). By combining $\tilde{F}_{\mathscr{A}}$ with the quotient homomorphism $C(\mathscr{A}) \to C(\mathscr{A})/\ker \partial_{\mathscr{A}}$ we obtain a homomorphism $F_{\mathscr{A}}\colon C(\mathscr{A}) \to R^+$. By construction $F_{\mathscr{A}}$ is trivial on $\ker \partial_{\mathscr{A}}$ and $F_{\mathscr{B}} = F_{\mathscr{A}} \circ i$. Lemma 2.5 shows that restricting $F_{\mathscr{A}}$ to generators provides the required ratio specification on $\mathscr{X}$.

Conversely, suppose $F_\alpha(x, y)$ is the restriction to $\mathscr{B}$ of a ratio specification on $\mathscr{X}$. We have the following commutative diagram:



Hence to show that $F_{\mathscr{B}}$ is trivial on $\ker \partial_{\mathscr{B}}$ it is sufficient to show that $F_{\mathscr{A}}$ is trivial on $\ker \partial_{\mathscr{A}}$. However, $F_{\mathscr{A}}$ comes from a ratio specification on $\mathscr{X}$, so by Lemma 2.5 it is trivial on $\ker \partial_{\mathscr{A}}$.

The next theorem describes all extensions of $F_{\mathscr{B}}$ to ratio specifications on $\mathscr{X}$. First we need

2.7. *Definition.* For $\mathscr{B} \subset \mathscr{A}$ we say that x *is connected to* $y \bmod \mathscr{B}$ if there exist $x_0, x_1, \ldots, x_n \in X_0, \beta_1, \ldots, \beta_n \in \mathscr{B}$ with $x_0 = x$, $x_n = y$ and $x_i = x_{i-1} \bmod \beta_i$, $1 \leqq i \leqq n$.

Being connected mod $\mathscr{B}$ is an equivalence relation on X_0; we shall write $\{x\}_{\mathscr{B}}$ for the class of elements connected to $x \bmod \mathscr{B}$. Note that the requirements on $\mathscr{A}$ imply that $x = y \bmod \mathscr{A}$ if and only if there exists $\alpha \in \mathscr{A}$ with $x = y \bmod \alpha$.

Choose a single representative from each $\mathscr{B}$ equivalence class and let the set of these chosen elements be denoted $X_{\mathscr{B}}$. We can choose representatives of the $\mathscr{A}$ equivalence classes so that $X_{\mathscr{A}} \subset X_{\mathscr{B}}$.

2.8. *Definition.* We denote by $\mathscr{G}$ the set of all positive real valued functions on $X_{\mathscr{B}}$. Two functions $f, g \in \mathscr{G}$ are called *equivalent* if $f(x)/f(y) = g(x)/g(y)$ for all $x, y \in X_{\mathscr{B}}$ with $x = y \bmod \mathscr{A}$. We use $\mathscr{H}$ to denote the equivalence classes of $\mathscr{G}$ under this relation.

2.9. **Theorem.** *Let $F_\alpha(x, y)$ be a positive real valued function defined for $\alpha \in \mathscr{B}$ and $x = y \bmod \alpha$. Suppose $F_{\mathscr{B}}$ defined by* (8) *is trivial on* $\ker \partial_{\mathscr{B}}$. *Then there is a one-to-one correspondence between $\mathscr{H}$ and extensions of $F_{\mathscr{B}}$ which come from ratio specifications on $\mathscr{X}$.*

Proof. Any extension of $F_{\mathscr{B}}$ to a ratio specification will satisfy certain conditions. In particular if $x = y \bmod \mathscr{B}$, then there is some $\alpha \in \mathscr{A}$ with $x = y \bmod \alpha$ and $F_\alpha(x, y)$ is uniquely determined. Thus we can assume without loss of generality that we are given $F_\alpha(x, y)$ satisfying (6) wherever $x = y \bmod \mathscr{B}$ and $x = y \bmod \alpha$. Now given $f \in \mathscr{G}$ and $(w, z, \alpha) \in C(\mathscr{A})$ we define

$$F_\alpha(w, z) = F_\gamma(w, x) F_\gamma(y, z) f(x)/f(y) \tag{9}$$

where $x, y \in X_{\mathscr{B}}$, $w = x \bmod \mathscr{B}$, $y = z \bmod \mathscr{B}$, $\alpha \subset \gamma$, $w = x \bmod \gamma$, $y = z \bmod \gamma$. Note that (9) is independent of the choice of γ satisfying the above. It is straightforward to verify that $F_\alpha(w, z)$ so defined satisfies (4), (5) and (6) and thus gives a ratio specification on $\mathscr{X}$ which extends that given. From the defining formula (9) it follows that equivalent elements of $\mathscr{G}$ yield the same ratio specification, while nonequivalent elements of $\mathscr{G}$ yield distinct ratio specifications.

Finally given a ratio specification $F_\alpha(x, y)$ we define $f \in \mathscr{G}$ by

$$f(x) = F_\alpha(x, y) \tag{10}$$

where $x \in X_{\mathscr{B}}$, $y \in X_{\mathscr{A}}$ and $x = y \bmod \alpha$. Since the y in $X_{\mathscr{A}}$ equivalent to x is unique and (6) holds, (10) is well defined. A calculation shows that the f so defined satisfies (9).

Note that $\mathscr{H}$ consists of a single element exactly when $\{x\}_{\mathscr{A}} = \{x\}_{\mathscr{B}}$ for each $x \in X_0$.

2.10. **Corollary.** *Assume $F_{\mathscr{B}}$ is trivial on* $\ker \partial_{\mathscr{B}}$. *Then the extension of $F_{\mathscr{B}}$ to a ratio specification on $\mathscr{X}$ is unique if and only if $\{x\}_{\mathscr{A}} = \{x\}_{\mathscr{B}}$ for all $x \in X_0$,* i.e. *if and only if whenever x, y are connected* mod $\mathscr{A}$, *they are connected* mod $\mathscr{B}$.

2.11. *Definition.* The *length* of the cycle $\sum_{i=1}^{n} k_i(z_i, y_i, \alpha_i) \in \ker \partial_{\mathscr{B}}$ is $\sum_{i=1}^{n} |k_i|$.

2.12. *Remark.* Let $\tilde{C}(\mathscr{B})$ be the free abelian group generated by pairs (x, y) where $x = y \bmod \alpha$ for $\alpha \in \mathscr{B}$. We have the mapping $\phi\colon C(\mathscr{B}) \to \tilde{C}(\mathscr{B})$ with

$$\phi\left(\sum_{i=1}^{n} k_i(x_i, y_i, \alpha_i)\right) = \sum_{i=1}^{n} k_i(x_i, y_i).$$

Thus we have the boundary operator $\tilde{\partial}_{\mathscr{B}}\colon \tilde{C}(\mathscr{B}) \to C(X_0)$ satisfying $\partial_{\mathscr{B}} = \tilde{\partial}_{\mathscr{B}} \circ \phi$. The homomorphism ϕ is onto and thus induces an isomorphism from $\ker \partial_{\mathscr{B}}/$

ker ϕ to ker $\tilde{\partial}_{\mathscr{B}}$. A set of generators for ker $\tilde{\partial}_{\mathscr{B}}$ thus provides a set of generators for ker $\partial_{\mathscr{B}}$/ker ϕ. We can pick representatives for these in ker $\partial_{\mathscr{B}}$ in such a way that the cycle length is preserved. These representatives, together with a generating set ker ϕ will generate ker $\partial_{\mathscr{B}}$. Finally we note that ker ϕ is generated by cycles of the form $(x, y, \alpha_1)-(x, y, \alpha_2)$.

3. Application to Random Fields

We now relate the algebraic formalism of the previous section to the model originally introduced. Recall that S is a countably infinite set, Y a finite set and $X = Y^S$. There is very little additional effort required to allow a different Y at each site, but for simplicity of notation we shall not do this. The set of allowed configurations for which we want our conditional probabilities positive is denoted X_0. In most cases considered in the literature X_0 is obtained from X by exclusion rules which involve sites at finite distances from each other. The set $\mathscr{A}$ is the set of all finite subsets of S. For $\alpha \in \mathscr{A}$ and $x, y \in X_0$, $x = y \bmod \alpha$ if $x^j = y^j$ for all $j \in S \setminus \alpha$. We use superscripts to denote components.

For a given X_0 and $\mathscr{B} \subset \mathscr{A}$ we wish to know whether conditional probabilities given for $\alpha \in \mathscr{B}$ determine the finite set conditional probabilities (i.e. those for $\mathscr{A}$) and a set of generators for ker $\partial_{\mathscr{B}}$ so we may express consistency conditions. For the case $X_0 = X$, if $\mathscr{B}$ contains all singletons $\{j\}$, $j \in S$, corresponding to one point conditional probabilities, then the finite set conditional probabilities can be computed (see [9]). Also cycles of length 4 are sufficient (see [9]). One needs, in addition, some regularity conditions; we shall express one form of these in a result below.

When S is a lattice in Euclidean space and X_0 is determined by finite range constraints, it can be quite a difficult combinatorial problem to determine, for a given $\mathscr{B}$, the connectedness and cycle structure. For systems with one dimensional geometry and constraints of finite range we can give a reasonable geometric expression of the algebraic criteria of the preceding section. By considering aggregates "along the line" the constraints can be considered to be nearest neighbour.

Specifically we consider the case in which $S = Z$, the integers, and Y is a finite set. We assume a function $M: Y \times Y \to R$ with $M(a, b) \geqq 0$. Then we define

$$X_0 = \{x \in Y^S : M(x^i, x^{i+1}) > 0 \text{ for all } i \in S\}.$$

Spaces of this type have received considerable study (see [12]). The one point conditional probabilities will, in general, not be sufficient to determine the finite set conditional probabilities. We shall show that under a certain condition the j-adjacent point conditional probabilities are sufficient.

3.1. **Theorem.** *Let $\mathscr{B}$ denote the collection of all subsets of S which consist of j adjacent integers. Assume the matrix M has the following property:*

$$M^{j+1}(a, c) > 0, \quad M(b, c) > 0 \Rightarrow M^j(a, b) > 0 \tag{11}$$

for all $a, b, c \in Y$. Then

(i) *If $x, y \in X_0$ are connected* mod $\mathscr{A}$, *they are connected* mod $\mathscr{B}$,

(ii) ker $\partial_{\mathscr{B}}$ *is generated by the set of cycles of length* $\leqq j+3$.

Proof. (i) Suppose $x, y \in X_0$ and $x = y \bmod \mathscr{A}$. Let $l(x, y) = n - m$ where m and n are respectively the first and last coordinates where x and y differ. If $l(x, y) \leqq j - 1$, then x and y differ by at most j adjacent coordinates so $x = y \bmod \mathscr{B}$. Otherwise, as $M^{j+1}(x^{n-j}, x^{n+1}) > 0$, $M(y^n, x^{n+1}) > 0$, we have $M^j(x^{n-j}, y^n) > 0$. Thus we can find $w_1, w_2, \ldots, w_{j-1} \in Y$ so that

$$z = (\ldots, x^{m-1}, x^m, \ldots, x^{n-j}, w_1, \ldots, w_{j-1}, y^n, x^{n+1}, \ldots)$$

is an element of X_0. Now x and z differ on at most j adjacent sites, and $l(z, y) < l(x, y)$. Iteration of the procedure at most $l(x, y) - j + 1$ times connects x to $y \bmod \mathscr{B}$.

(ii) By Remark 2.12 it is sufficient to show that $\ker \tilde{\partial}_{\mathscr{B}}$ is generated by cycles of length $\leqq j + 3$. Now since $\ker \tilde{\partial}_{\mathscr{B}}$ is generated by cycles of the form $c = \sum_{i=1}^{n} (x_i, x_{i+1})$ where $x_i \in X_0$, $1 \leqq i \leqq n$; $x_1 = x_{n+1}$ and x_i, x_{i+1} differ on at most j adjacent sites, it suffices to prove the result for such cycles. For $x, y \in X_0$, $x = y \bmod \mathscr{A}$ let

$$\begin{aligned}
F(x, y) &= \text{the first site where } x \text{ and } y \text{ differ;} \\
T(x, y) &= \text{the last site where } x \text{ and } y \text{ differ;} \\
f(c) &= \min_{1 \leqq i \leqq n} F(x_i, x_{i+1}); \\
t(c) &= \max_{1 \leqq i \leqq n} T(x_i, x_{i+1}); \\
P(c) &= \text{least } i \text{ for which } f(c) = F(x_i, x_{i+1}); \\
Q(c) &= \text{largest } i \text{ for which } f(c) = F(x_i, x_{i+1}).
\end{aligned}$$

The aim is to write c as a sum of cycles of length $\leqq j + 3$ plus a cycle d with

$$f(c) < f(d) \leqq t(d) \leqq t(c). \tag{12}$$

After a finite number of iterations of this procedure we have c expressed as the sum of cycles of length $\leqq j + 3$.

First we consider the case in which $t(c) - f(c) \leqq j - 1$. Then each pair (x_1, x_k), $2 \leqq k \leqq n$ is equivalent mod α for some $\alpha \in \mathscr{B}$ so

$$c = \sum_{i=2}^{n-1} \{(x_1, x_i) + (x_i, x_{i+1}) + (x_{i+1}, x_1)\}$$

expresses c as the sum of 3 cycles.

When $t(c) - f(c) \geqq j$ we proceed to reduce this difference in two stages. We have $P(c) < Q(c)$ and $x_i^{f(c)} = x_1^{f(c)}$ for $1 \leqq i \leqq P(c)$ or $Q(c) < i \leqq n$, since the least site which changes must eventually return to the original value. The first stage is to write c as the sum of a cycle of length $j + 3$ or less and a cycle d with $Q(d) - P(d) < Q(c) - P(c)$. We repeat this until $Q(d) - P(d) = 1$. The second stage is to express a cycle c with $Q(c) - P(c) = 1$ as the sum of a 3 cycle and a cycle d satisfying (12).

We now consider this second stage, *i.e.* $t(c) - f(c) \geqq j$ and $Q(c) - P(c) = 1$. There is no loss of generality in assuming $P(c) = 1$. Then

$$c=\{(x_1,x_2)+(x_2,x_3)+(x_3,x_1)\}+(x_1,x_3)+\sum_{i=3}^{n}(x_i,x_{i+1})$$

gives the required representation, since x_1 and x_3 differ at most on j-adjacent sites. This completes stage two.

For stage one, *i.e.* $Q(c)-P(c)>1$, we have two cases to consider. For simplicity of notation we assume that $f(c)=0$.

Case (*a*). $F(x_2,x_3)\geqq j+1$.
Set $z=(\ldots,x_1^{-1},x_1^0,\ldots,x_1^j,x_3^{j+1},x_3^{j+2},\ldots)$.
Then $z\in X_0$ and there exist $\alpha,\beta\in\mathscr{B}$ with $z=x_1 \bmod \alpha$, $z=x_2 \bmod \beta$. Then

$$d=(x_1,z)+(z,x_3)+\sum_{i=3}^{n}(x_i,x_{i+1})$$

satisfies $Q(d)-P(d)<Q(c)-P(c)$ and

$$c=d+(x_1,x_2)+(x_2,x_3)+(x_3,z)+(z,x_1),$$

i.e. c is the sum of d and *a* four cycle.

Case (*b*). $F(x_2,x_3)\leqq j$.
By the method of proof of part (i) we can find $z_1,\ldots,z_k\in X_0$ with $z_1=x_1$, $z_k=x_3$ and z_i differing from z_{i+1} at most on j-adjacent sites. Also $F(z_i,z_{i+1})>0$ for $1\leqq i<k-1$ and $T(z_i,z_{i+1})\leqq t(c)$ for $1\leqq i\leqq k$. We can do this with $2\leqq k\leqq F(x_2,x_3)+2$. Then with

$$d=\sum_{i=1}^{k-1}(z_i,z_{i+1})+\sum_{i=3}^{n}(x_i,x_{i+1})$$

we have $c=d+(x_1,x_2)+(x_2,z_k)+(z_k,z_{k-1})+\ldots+(z_2,z_1)$.

So c can be expressed as the sum of d and a cycle of length $k+1\leqq j+3$. For this d we have $Q(d)-P(d)<Q(c)-P(c)$. This completes the proof.

3.2. *Remark.* Essentially the same proof can be carried out when S is the positive integers instead of all integers.

3.3. *Remark.* Condition (11) of Theorem 3.1 can be replaced by

$$M^{j+1}(a,c)>0, \qquad M(a,b)>0\Rightarrow M^j(b,c)>0, \tag{13}$$

with the proof simply reversing the order of certain operations. Any homogeneous finite Markov chain without transient states will satisfy conditions (11) and (13) for sufficiently large j. These conditions and the proof can be adapted to inhomogeneous Markov chains with state spaces varying from site to site.

3.4. *Example.* Let Y be the set of j digit numbers in an arbitrary fixed integer base. Define $M([d_1d_2\ldots d_j],[d_2d_3\ldots d_je])=1$ and $M(a,b)=0$ otherwise. Since $M^j(a,b)=1$ for all $a,b\in Y$, Theorem 3.1 shows that a ratio specification is uniquely determined by its $\mathscr{B}$ values, with $\mathscr{B}$ the collection of j-adjacent point subsets of S. Two distinct elements of X_0 must differ by at least j sites so knowledge of the ratio specification for sets with $j-1$ and fewer elements gives no information about the ratio specification for other sets; the $j-1$ point conditional probabilities are trivial.

We now give an illustration of how the algebraic techniques above can be applied in terms of actual conditional probabilities. We use the notation of Theorem 3.1. The topology of $X = Y^S$ is explained in [9], to which we refer the reader for an explanation of the notation $\mu(\omega = x \text{ on } \Lambda \mid \omega = y \text{ on } \Lambda^c)$. X_0 is a closed subspace of X with the subspace topology.

3.5. **Theorem.** *Let the hypothesis of Theorem* 3.1 *be satisfied. Assume that the real valued continuous function $P_\alpha(x, y)$ is given for each $\alpha \in \mathscr{B}$ and $x = y \bmod \alpha$ which satisfies* (1) *and* (2) *and* $\sum_x P_\alpha(x, y) = 1$ *for each $\alpha \in \mathscr{B}$ and $y \in X_0$, with the sum over those x which are equivalent to $y \bmod \alpha$. Define $F_\alpha(x, y) = P_\alpha(x, y)/P_\alpha(y, x)$ and $F_{\mathscr{B}}$ on $C(\mathscr{B})$ by* (8). *Assume $F_{\mathscr{B}}$ is trivial on all elements of $\ker \partial_{\mathscr{B}}$ of length $\leq j+3$. If X_0 is nonempty, then there is a probability measure μ on X_0 such that*

$$\mu(\omega = x \text{ on } \Lambda \mid \omega = y \text{ on } \Lambda^c) = P_\Lambda(x, y) \qquad \mu. \text{ a.e.}$$

for each $\Lambda \in \mathscr{B}$ and $x = y \bmod \Lambda$.

Proof. By Theorems 2.6 and 3.1 $F_{\mathscr{B}}$ has a unique extension to a ratio specification on $\mathscr{X}$. By Lemma 2.4 we have an equivalence class of algebraic specifications corresponding to $F_{\mathscr{B}}$. By the requirement that $\sum_x P_\alpha(x, y) = 1$ we have a uniquely defined algebraic specification on $\mathscr{X}$ corresponding to $F_{\mathscr{B}}$ which coincides for $\alpha \in \mathscr{B}$ with that originally given. We have continuity for $P_\alpha(x, y)$ since only a finite number of elementary operations are needed to compute it from the given values. By Lemma 5.1 of [7], the $P_\alpha(x, y)$ are consistent. The existence of the required μ follows from Theorem 3.1 of [7].

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Generalised Holley-Preston Inequalities on Measure Spaces and Their Products

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Summary. It is shown that if $(X, \mathscr{F}, \mu)$ is a product of totally ordered measure spaces and f_j $(j=1,2,3,4)$ are measurable non-negative functions on X satisfying

$$f_1(x)f_2(y) \leqq f_3(x \vee y)f_4(x \wedge y),$$

where $(\vee, \wedge)$ are the lattice operations on X, then

$$(\int f_1 \, d\mu)(\int f_2 \, d\mu) \leqq (\int f_3 \, d\mu)(\int f_4 \, d\mu).$$

This generalises results of Ahlswede and Daykin (for counting measure on finite sets) and Preston (for special choices of f_j).

1. Introduction

In recent years a number of inequalities have been discovered relating the cardinalities of subsets of a finite distributive lattice L and also the values of certain functions on L satisfying special conditions. One of the first of these was obtained by Kleitman [12] who showed that if S is any finite set, U an up-set and D a down-set in the lattice $L=2^S$ of all subsets of S, then

$$|L||U \cap D| \leqq |U||D| \tag{1.1}$$

where $|A|$ denotes the cardinality of A. This was strengthened by Daykin [5] who showed that for any subsets F and G of any finite distributive lattice L,

$$|F||G| \leqq |F \vee G||F \wedge G| \tag{1.2}$$

where

$$F \vee G = \{x \vee y : x \in F, y \in G\}, \quad F \wedge G = \{x \wedge y : x \in F, y \in G\}.$$

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This result shows that (1.1) is valid in any finite distirbutive lattice, and also implies several other previously known inequalities. Daykin [6] later considered functions $A \to F_A$ and $A \to G_A$ of 2^S into 2^L, where S is any finite set and L is any finite distributive lattice. He showed that

$$\sum_{A \subset S} |F_A| |G_{S \setminus A}| \leqq \sum_{C \subset S} |H_C| |K_C| \tag{1.3}$$

where

$$H_C = \bigcup_{A \cup B = C} F_A \vee G_B, \quad K_C = \bigcup_{A \cup B = C} F_{S \setminus A} \wedge G_{S \setminus B}.$$

When S is empty, (1.3) reduces to (1.2). If we put

$$f_1(A) = |F_A| \quad f_2(A) = |G_A|$$
$$f_3(A) = |H_A| \quad f_4(A) = |K_{S \setminus A}|$$

then (1.2) gives

$$f_1(A) f_2(B) \leqq f_3(A \cup B) f_4(A \cap B). \tag{1.4}$$

More general non-negative functions satisfying inequalities similar to (1.4) had already been considered by a number of authors. Fortuin, Kastelyn and Ginibre [9] showed that if f_1, f_2, f_3 and f_4 coincide on 2^S and satisfy (1.4), then

$$\left(\sum_{A \subset S} u(A) f_1(A)\right)\left(\sum_{B \subset S} v(B) f_2(B)\right) \leqq \left(\sum_{C \subset S} u(C) v(C) f_3(C)\right)\left(\sum_{D \subset S} f_4(D)\right) \tag{1.5}$$

for any increasing (non-negative) functions u and v on 2^S. Holley [10] extended this by showing that if $f_1 = f_3$, $f_2 = f_4$ and (f_1, f_2, f_3, f_4) satisfies (1.4), then

$$\left(\sum_{A \subset S} f_1(A)\right)\left(\sum_{B \subset S} v(B) f_2(B)\right) \leqq \left(\sum_{C \subset S} v(C) f_3(C)\right)\left(\sum_{D \subset S} f_4(D)\right). \tag{1.6}$$

(At first sight (1.6) may appear less general than (1.5) but this is only because of the stronger conditions imposed in [9]. To obtain (1.5) as a special case of (1.6) one simply replaces f_1 by uf_1 and f_3 by uf_3, noting that (1.4) is still valid.) A further extension was made by Preston [13] who showed that if $(X, \mathscr{F}, \mu)$ is a finite product of totally ordered measure spaces (so X is a lattice in the product ordering), f_1 and f_2 are non-negative integrable functions on X, v_1 is a bounded increasing measurable function on X, $f_3 = f_1$, $f_4 = f_2$, $v_2 = v_1$ and (f_1, f_2, f_3, f_4) satisfy the lattice-theoretic analogue of (1.4):

$$f_1(x) f_2(y) \leqq f_3(x \vee y) f_4(x \wedge y) \tag{1.7}$$

then

$$\left(\int f_1 \, d\mu\right)\left(\int v_1 f_2 \, d\mu\right) \leqq \left(\int v_2 f_3 \, d\mu\right)\left(\int f_4 \, d\mu\right). \tag{1.8}$$

The case of counting measure on a finite set gave (1.6). Preston's theorem was extended to countable products of probability spaces by the first author [3]. A simplified proof of (1.8) in finite products was found independently by Edwards [8] and Kemperman [11] (Edwards also considered countable products). However this method did not show the existence of certain interesting measures on

X^2 as established in [10, 13]. The original applications of (1.5) were to problems in statistical mechanics, but Seymour and Welsh [14, 15] have shown how (1.5), (1.6) and (1.8) are important in combinatorial and percolation theory.

Edwards and Kemperman in fact found that the inductive proof of (1.8) was made simpler if they replaced the assumption that v_1 and v_2 coincide and are increasing by the weaker condition:

$$v_1(x) \leqq v_2(y) \quad \text{whenever } x \leqq y \tag{1.9}$$

but they retained the conditions $f_3 = f_1$ and $f_4 = f_2$. Ahlswede and Daykin [1] dropped even these conditions, thereby making it unnecessary to introduce v_1 and v_2 at all, but they considered only finite sets. They showed that if f_1, f_2, f_3 and f_4 are non-negative functions satisfying (1.7) on a finite distributive lattice L then

$$(\sum_{x \in L} f_1(x))(\sum_{y \in L} f_2(y)) \leqq (\sum_{z \in L} f_3(z))(\sum_{w \in L} f_4(w)). \tag{1.10}$$

In [2] they considered more abstract situations in which the lattice L and operations $\vee$ and $\wedge$ are replaced by any finite set S and mappings ϕ and ψ of S^2 into S. Defining (ϕ, ψ) to be "$\mathfrak{M}$-expansive" if

$$(\sum_{x \in S} f_1(x))(\sum_{y \in S} f_2(y)) \leqq (\sum_{z \in S} f_3(z))(\sum_{w \in S} f_4(w)) \tag{1.11}$$

whenever

$$f_1(x) f_2(y) \leqq f_3(\phi(x, y)) f_4(\psi(x, y)), \tag{1.12}$$

they showed that the class of $\mathfrak{M}$-expansive pairs of mappings (ϕ, ψ) is closed under direct products. While studying direct products, they were led to introduce an apparently stronger notion of "$\mathfrak{M}$-explosiveness", which they showed to be equivalent to $\mathfrak{M}$-expansiveness.

The aim of this paper is to exhibit a single inequality which includes (1.1), (1.2), (1.5), (1.6), (1.8) and (1.10) as special cases. Thus it will be shown that if $(X, \mathscr{F}, \mu)$ is a finite product of totally ordered measure spaces and f_1, f_2, f_3 and f_4 are measurable functions of X into $[0, \infty]$ satisfying (1.7), then

$$(\int f_1 \, d\mu)(\int f_2 \, d\mu) \leqq (\int f_3 \, d\mu)(\int f_4 \, d\mu). \tag{1.13}$$

At this level of generality the inductive step in the proof becomes much simpler than that in [10] or [13]. Furthermore the extension to countably infinite products is now a straightforward application of the Fubini-Jessen theorem. The main difficulty in proving the infinite case in [3] and [8] was caused by uncertainty as to whether (1.8) is satisfied if (1.7) holds only μ^2-a.e. in X^2. We shall see here that in finite products this is the case, and indeed (1.13) is always valid if (1.7) holds μ^2-a.e.

The main results are presented for abstract pairs (ϕ, ψ) rather than $(\vee, \wedge)$, and they therefore include also (1.11) as a special case. Section 2 of the paper is devoted to a study of the inequality (1.12), Sect. 3 contains the main results showing that certain pairings (ϕ, ψ) satisfy the measure-theoretic analogue of $\mathfrak{M}$-expansiveness, and Sect. 4 contains a discussion of a measure-theoretic version of

$\mathfrak{M}$-explosiveness. The final section is devoted to finite distributive lattices, showing how inequalities such as (1.1), (1.2) and (1.3) are related to $\mathfrak{M}$-expansiveness and to each other.

It will be convenient to allow funcitons to take values in the extended non-negative reals $[0, \infty]$, and we shall adopt the usual conventions concerning the arithmetic of this system, except that we shall regard $0.\infty$ and $\infty.0$ as undefined. Furthermore an inequality will be considered to be satisfied if either side is undefined. We shall also need to consider n-tuples of functions taking values in $[0, \infty]$, and we shall denote these interchangably either componentwise as $(f_1, \ldots, f_n)$ or as a single function $\mathbf{f}: X \to [0, \infty]^n$.

2. Compatibility

Our basic object of study will be a system $\mathscr{S} = (X, \mathscr{F}, \mu, \phi, \psi)$ consisting of a σ-finite measure space $(X, \mathscr{F}, \mu)$, whose measure-theoretic product with itself will be denoted by $(X^2, \mathscr{F}^2, \mu^2)$, together with a mapping (ϕ, ψ) of X^2 into itself. Such a system will be called a *paired measure space* with *pairing* (ϕ, ψ). If (ϕ, ψ) is $\mathscr{F}^2$-measurable, we shall say that $\mathscr{S}$ is *measurably paired*. As a matter of notational convenience, the component parts of a paired measure space denoted by $\mathscr{S}_\lambda$, where λ may be an index, will themselves always be denoted by $(X_\lambda, \mathscr{F}_\lambda, \mu_\lambda, \phi_\lambda, \psi_\lambda)$.

An *involution* on $\mathscr{S}$ is a bijection π of X such that for any x and y in X and E in $\mathscr{F}$, $\pi(\pi(x)) = x$, $\phi(\pi(x), \pi(y)) = \pi(\psi(y, x))$, $\pi(E) \in \mathscr{F}$ and $\mu(\pi(E)) = \mu(E)$. Following [2], an $\mathscr{F}$-measurable function $\mathbf{f}$ of X into $[0, \infty]^4$ will be said to be *compatible* (resp. *μ-compatible*, resp. *diagonally μ-compatible*) with (ϕ, ψ) if

$$f_1(x) f_2(y) \leqq f_3(\phi(x, y)) f_4(\psi(x, y)) \tag{2.1}$$

for all (resp. μ^2-almost all, resp. $\bar{\mu}$-almost all) pairs (x, y) in X^2, where $\bar{\mu}$ is the image of μ under the mapping $x \to (x, x)$ of X onto the diagonal Δ in X^2. Thus $\mathbf{f}$ is diagonally μ-compatible if

$$f_1(x) f_2(x) \leqq f_3(\phi(x, x)) f_4(\psi(x, x)) \tag{2.2}$$

μ-a.e.(x). The sets of all compatible, μ-compatible and diagonally μ-compatible functions $\mathbf{f}$ will be denoted by $\mathfrak{K}(\phi, \psi)$, $\mathfrak{K}_\mu(\phi, \psi)$ and $\mathfrak{K}_\mu^d(\phi, \psi)$ respectively, or simply by $\mathfrak{K}$, $\mathfrak{K}_\mu$ and $\mathfrak{K}_\mu^d$ if no confusion is likely. A 4-tuple (E_1, E_2, E_3, E_4) of $\mathscr{F}$-measurable sets will be said to be *compatible* with (ϕ, ψ) if

$$\phi(E_1 \times E_2) \subset E_3 \quad \text{and} \quad \psi(E_1 \times E_2) \subset E_4 \tag{2.3}$$

or equivalently if their characteristic functions $(\chi_{E_1}, \chi_{E_2}, \chi_{E_3}, \chi_{E_4})$ are compatible. Thus the set of compatible 4-tuples of sets is in one-to-one correspondence with the set $\mathfrak{P}$ of functions in $\mathfrak{K}$ taking values in $\{0, 1\}^4$.

For (x, y) in X^2, put $(x, y)^* = (y, x)$. For any function α on X^2, put $\alpha^*(z) = \alpha(z^*)$. The following properties of compatibility are clear:

$$\mathbf{f} \in \mathfrak{K}(\phi, \psi) \Leftrightarrow (f_2, f_1, f_3, f_4) \in \mathfrak{K}(\phi^*, \psi^*) \tag{2.4}$$

$$\mathbf{f} \in \mathfrak{K}(\phi, \psi) \Leftrightarrow (f_1, f_2, f_4, f_3) \in \mathfrak{K}(\psi, \phi) \tag{2.5}$$

$$\mathbf{f}, \mathbf{g} \in \mathfrak{R}(\phi, \psi) \Rightarrow (f_1 \cdot g_1, f_2 \cdot g_2, f_3 \cdot g_3, f_4 \cdot g_4) \in \mathfrak{R}(\phi, \psi) \tag{2.6}$$

$$\mathbf{f} \in \mathfrak{R}(\phi, \psi) \Leftrightarrow (f_1 \circ \pi, f_2 \circ \pi, f_3 \circ \pi, f_4 \circ \pi) \in \mathfrak{R}(\psi^*, \phi^*) \tag{2.7}$$

where π is an involution.

Suppose that Y is an $\mathscr{F}$-measurable subset of X such that $\phi(Y^2) \subset Y$ and $\psi(Y^2) \subset Y$. Let $\mathscr{S}|_Y = (Y, \mathscr{F}_Y, \mu_Y, \phi_Y, \psi_Y)$ be the paired measure space obtained from $\mathscr{S}$ by restriction to Y. Given $\mathbf{g}$: $Y \to [0, \infty]^4$, let $\mathbf{g}_X$ be the extension of $\mathbf{g}$ to X vanishing outside Y. Then

$$\mathbf{g} \in \mathfrak{R}(\phi_Y, \psi_Y) \Leftrightarrow \mathbf{g}_X \in \mathfrak{R}(\phi, \psi). \tag{2.8}$$

Both μ-compatibility and diagonal μ-compatibility have properties similar to (2.4), (2.5), (2.6), (2.7) and (2.8).

In general there is no reason to suppose that μ-compatibility implies diagonal μ-compatibility. However the technical lemma in [3] gave one particular circumstance in which this phenomenon does occur, and we shall be deeply involved with the property in Sect. 3. Thus we shall say that $\mathscr{S}$ is *diagonally settled* if $\mathfrak{R}_\mu$ is contained in $\mathfrak{R}_\mu^d$. If μ is purely atomic, then $\bar{\mu}$ is absolutely continuous with respect to μ^2, and $\mathscr{S}$ is diagonally settled. The following two propositions in this section give some other examples of diagonally settled spaces.

The fundamental example to be considered at this stage occurs when X is a lattice and ϕ and ψ are the lattice operations $\vee$ and $\wedge$ (cf. [1, 5, 13]). Then X is totally ordered if and only if $\{x \vee y, x \wedge y\} = \{x, y\}$ for all (x, y) in X^2. In general we shall say that the pairing (ϕ, ψ) is *selective*, and $\mathscr{S}$ is *selectively paired*, if $\{\phi(x, y), \psi(x, y)\} = \{x, y\}$ for all (x, y) in X^2. (In this case, ϕ is an arbitrary choice from each ordered pair (x, y), and ψ is the other choice.) We shall also say that $\mathscr{S}$ is *diagonally invariant* if $\phi(x, x) = \psi(x, x) = x$ for all x in X. Clearly any selectively paired space is diagonally invariant.

Proposition 2.1. *Any selectively paired measure space $\mathscr{S}$ is diagonally settled.*

Proof. Consider a μ-compatible function $\mathbf{f}$: $X \to [0, \infty]^4$, and let

$$X_0 = \{y \in X: (2.1) \text{ holds for } \mu\text{-almost all } x\}.$$

Then $X \setminus X_0$ is μ-null, so replacing $\mathscr{S}$ by $\mathscr{S}|_{X \setminus X_0}$, we may assume that $X = X_0$.

If $f_1(x) = 0$, $f_2(x) = 0$, $f_3(x) = \infty$ or $f_4(x) = \infty$, then (2.2) is automatically satisfied. Thus we can assume none of these possibilities occurs in X. Let $X_1 = \{x \in X: f_4(x) = 0\}$. Then $f_1(x) f_2(y) = 0$ μ^2-a.e. (x, y) in X_1^2. Hence either f_1 or f_2 vanishes μ-a.e. in X_1, so X_1 is null. Thus replacing $\mathscr{S}$ by $\mathscr{S}|_{X \setminus X_1}$, we can assume that $f_4(x) > 0$ and similarly that $f_3(x) > 0$, $f_1(x) < \infty$ and $f_2(x) < \infty$.

Put $g(x) = f_2(x) f_3(x)^{-1}$ and $h(x) = f_2(x) f_4(x)^{-1}$, so that $0 < g(x) < \infty$ and $0 < h(x) < \infty$. Then (2.1) shows that for fixed y,

$$f_1(x)\, g(y) \leqq f_4(x) \tag{2.9}$$

for μ-almost all x with $\phi(x, y) = y$, and

$$f_1(x)\, h(y) \leqq f_3(x) \tag{2.10}$$

for μ-almost all x with $\phi(x,y)=x$. For integers $k,n\geq 1$, let $E_{kn}=g^{-1}((k-1)2^{-n}, k2^{-n}]$, and, assuming that E_{kn} is non-empty, choose a sequence y_{kn}^r $(r=1,2,\ldots)$ in E_{kn} such that

$$\sup\{h(y_{kn}^r): r=1,2,\ldots\}=\sup\{h(x): x\in E_{kn}\}. \tag{2.11}$$

Let $E'_{kn}=\{x\in E_{kn}: \phi(x,y_{kn}^r)=x \text{ for all } r\}$ and $E'=\bigcup_{k,n} E'_{kn}$. Applying (2.10), we see that for μ-almost all x in E'_{kn},

$$f_1(x)\,h(y_{kn}^r)\leqq f_3(x),$$

so it follows from (2.11) that $f_1(x)\,h(x)\leqq f_3(x)$, i.e. (2.2) holds μ-a.e. in E'.

Now consider x in $X\setminus E'$. For each n, there is a (unique) integer k such that x belongs to $E_{kn}\setminus E'_{kn}$. For some r, $\phi(x,y_{kn}^r)=y_{kn}^r$. It follows from (2.9) that for μ-almost all such x,

$$f_1(x)(g(x)-2^{-n})\leqq f_1(x)\,g(y_{kn}^r)\leqq f_4(x).$$

Letting $n\to\infty$, it now follows that $f_1(x)\,g(x)\leqq f_4(x)$, i.e. (2.2) holds. Thus $\mathbf{f}$ is diagonally μ-compatible.

We begin now our consideration of product spaces. The *direct product* of a finite family of paired measure spaces $\{\mathscr{S}_\lambda: \lambda\in\Lambda\}$, or an infinite family of paired probability spaces, is defined to be the paired measure space $\mathscr{S}=(X,\mathscr{F},\mu,\phi,\psi)$ where $(X,\mathscr{F},\mu)$ is the measure-theoretic product of $\{(X_\lambda,\mathscr{F}_\lambda,\mu_\lambda)\}$ and ϕ and ψ are defined by:

$$\phi(x,y)=(\phi_\lambda(x_\lambda,y_\lambda)),\qquad \psi(x,y)=(\psi_\lambda(x_\lambda,y_\lambda)).$$

In the case when $\Lambda=\{1,2\}$, we may denote $\mathscr{S}$, ϕ and ψ by $\mathscr{S}_1\times\mathscr{S}_2$, $\phi_1\times\phi_2$ and $\psi_1\times\psi_2$.

Proposition 2.2. *A finite direct product of diagonally settled measurably paired measure spaces is diagonally settled and measurably paired.*

Proof. It suffices by induction to consider the case when $\mathscr{S}=\mathscr{S}_1\times\mathscr{S}_2$. The measurability is clear. Take a function $\mathbf{f}$ which is μ-compatible with (ϕ,ψ). For μ_2^2-almost all (x_2,y_2) in X_2^2, the following inequality is valid μ_1^2-a.e. (x_1,y_1):

$$f_1(x_1,x_2)f_2(y_1,y_2)\leqq f_3(\phi_1(x_1,y_1),\phi_2(x_2,y_2))f_4(\psi_1(x_1,y_1),\psi_2(x_2,y_2)). \tag{2.12}$$

Thus if we put

$$\begin{aligned} h_1(x_1)&=f_1(x_1,x_2) & h_2(x_1)&=f_2(x_1,y_2)\\ h_3(x_1)&=f_3(x_1,\phi_2(x_2,y_2)) & h_4(x_1)&=f_4(x_1,\psi_2(x_2,y_2)) \end{aligned} \tag{2.13}$$

then $\mathbf{h}$ is μ_1-compatible and hence diagonally μ_1-compatible with (ϕ_1,ψ_1), so that the following inequality is valid for μ_1-almost all x_1 in X_1:

$$f_1(x_1,x_2)f_2(x_1,y_2)\leqq f_3(\phi_1(x_1,x_1),\phi_2(x_2,y_2))f_4(\psi_1(x_1,x_1),\psi_2(x_2,y_2)). \tag{2.14}$$

Hence for μ_1-almost all x_1, (2.14) holds μ_2^2-a.e. (x_2,y_2). (The measurability of (ϕ_j,ψ_j) is used here to allow Tonelli's form of Fubini's theorem to be applied.) A similar argument now shows that (2.14) holds $\bar{\mu}_2$-a.e. Hence $\mathbf{f}$ is diagonally μ-compatible with (ϕ,ψ).

Example 2.3. Let $\mathscr{S}$ be the direct product of a sequence of measurably paired probability spaces $\mathscr{S}_n$, and suppose that there exist $\mathscr{F}_n$-measurable sets E_n such that

$$\sum_n \mu_n(E_n) = \infty$$

$$\sum_n \mu_n^2(\phi_n^{-1}(E_n)) < \infty.$$

Let $E = \{x \in X\colon x_n \in E_n \text{ for infinitely many } n\}$. By the Borel-Cantelli lemmas, $\mu(E) = 1$ and $\mu^2(\phi^{-1}(E)) = 0$. Thus the function $(1, 1, 1 - \chi_E, 1)$ is μ-compatible. However if each $\mathscr{S}_n$ and hence $\mathscr{S}$ is diagonally invariant, this function is not diagonally μ-compatible, so $\mathscr{S}$ is not diagonally settled.

As a specific example, we can take $\mathscr{S}_n$ to be the unit interval $[0, 1]$ equipped with Lebesgue measure and the lattice operations, and $E_n = [0, n^{-1}]$.

3. Expansiveness

We shall now introduce the concept of expansiveness as a generalisation of the set-theoretic ideas developed in [2]. A paired measure space $\mathscr{S}$ will be said to be *expansive* if

$$\mu(E_1)\mu(E_2) \leqq \mu(E_3)\mu(E_4) \tag{3.1}$$

for all 4-tuples (E_1, E_2, E_3, E_4) of sets compatible with (ϕ, ψ); $\mathfrak{M}$-*expansive* if

$$\mu(f_1)\mu(f_2) \leqq \mu(f_3)\mu(f_4) \tag{3.2}$$

for all compatible functions $\mathbf{f}$ (where $\mu(f_j)$ is written in place of $\int_X f_j d\mu$); *strongly* $\mathfrak{M}$-*expansive* if (3.2) holds for all μ-compatible $\mathbf{f}$.

We begin by making some observations concerning these definitions.

Remarks 3.1. (a) The inequality (3.2) is automatically satisfied if either f_3 or f_4 is not μ-integrable, while if they are both integrable, and neither f_1 nor f_2 is μ-null, (3.2) requires that f_1 and f_2 be integrable.

(b) It is clear from (2.3) that $\mathscr{S}$ is expansive if and only if

$$\mu(E_1)\mu(E_2) \leqq \mu^*(\phi(E_1 \times E_2))\,\mu^*(\psi(E_1 \times E_2))$$

for all E_1 and E_2 in $\mathscr{F}$, where μ^* denotes outer μ-measure. In particular, in an expansive probability space,

$$\mu^*(\phi(X^2)) = \mu^*(\psi(X^2)) = 1.$$

(c) If $\mathscr{S}$ is expansive (resp. $\mathfrak{M}$-expansive, strongly $\mathfrak{M}$-expansive), and Y is an $\mathscr{F}$-measurable subset of X with $\phi(Y^2) \subset Y$ and $\psi(Y^2) \subset Y$, then it follows from (2.8) that the restricted paired measure space $\mathscr{S}|_Y$ is expansive (resp. $\mathfrak{M}$-expansive, strongly $\mathfrak{M}$-expansive).

(d) It follows from (2.6) that for (E_1, E_2, E_3, E_4) and (f_1, f_2, f_3, f_4) compatible (or μ-compatible) with (ϕ, ψ), the function $(f_1\chi_{E_1}, f_2\chi_{E_2}, f_3\chi_{E_3}, f_4\chi_{E_4})$ is also

compatible (or μ-compatible). Thus in an $\mathfrak{M}$-expansive (or strongly $\mathfrak{M}$-expansive) space,

$$(\int_{E_1} f_1\,d\mu)(\int_{E_2} f_2\,d\mu) \leqq (\int_{E_3} f_3\,d\mu)(\int_{E_4} f_4\,d\mu). \tag{3.3}$$

In particular if S is a finite set, μ_c is counting measure and (ϕ, ψ) is a mapping of S^2 into itself, then (ϕ, ψ) is expansive (resp. $\mathfrak{M}$-expansive) in the sense defined in [2, pp. 268–269] if and only if $(S, 2^S, \mu_c, \phi, \psi)$ is expansive (resp. $\mathfrak{M}$-expansive or equivalently strongly $\mathfrak{M}$-expansive) according to the above definitions. In this respect the following study of expansive measure spaces extends the detailed investigation carried out in [2]. The theorem and corollary in [1] now become the statements that if $S = 2^{\{1,\ldots,n\}}$ then $(S, 2^S, \mu_c, \cup, \cap)$ is (strongly) $\mathfrak{M}$-expansive, and more generally that if S is a distributive lattice, then $(S, 2^S, \mu_c, \vee, \wedge)$ is (strongly) $\mathfrak{M}$-expansive. The reader is referred to [2] for some examples of expansive pairings on finite sets.

(e) Following [2], it would be possible to define $\mathfrak{C}$-expansiveness for an arbitrary subset $\mathfrak{C}$ of $\mathfrak{R}_\mu$ by requiring that (3.2) should hold for all $\mathbf{f}$ in $\mathfrak{C}$. Expansiveness, $\mathfrak{M}$-expansiveness and strong $\mathfrak{M}$-expansiveness correspond respectively to taking $\mathfrak{C} = \mathfrak{P}$, $\mathfrak{R}$ and $\mathfrak{R}_\mu$. Other interesting classes $\mathfrak{C}$ might include the bounded and the integrable functions in $\mathfrak{R}$ and in $\mathfrak{R}_\mu$ (note the comments on integrability in (a) above), and many of our results remain valid for such classes. However we shall not list these as it should be clear from the proofs under what circumstances these variations are valid.

Proposition 3.2. *Let π be an involution on a (strongly) $\mathfrak{M}$-expansive measure space $\mathscr{S}$. Then for any (μ-)compatible* $\mathbf{f}$,

$$\mu(f_1 \cdot (f_2 \circ \pi)) \leqq \mu(f_3 \cdot (f_4 \circ \pi)). \tag{3.4}$$

Proof. It follows from (2.4)–(2.7) that $(f_1 \cdot (f_2 \circ \pi), f_2 \cdot (f_1 \circ \pi), f_3 \cdot (f_4 \circ \pi), f_4 \cdot (f_3 \circ \pi))$ is (μ-)compatible. Since π is measure-preserving, (3.2) gives

$$\begin{aligned} \mu(f_1 \cdot (f_2 \circ \pi))^2 &= \mu(f_1 \cdot (f_2 \circ \pi))\,\mu(f_2 \cdot (f_1 \circ \pi)) \\ &\leqq \mu(f_3 \cdot (f_4 \circ \pi))\,\mu(f_4 \cdot (f_3 \circ \pi)) \\ &= \mu(f_3 \cdot (f_4 \circ \pi))^2. \end{aligned}$$

Example 3.3. Consider the case when $X = \{x, y\}$, $\mathscr{F} = 2^X$, $\mu = \mu_c$,

$$\phi(x,x) = \phi(x,y) = \phi(y,x) = x, \quad \phi(y,y) = y$$

$$\psi(x,x) = x, \quad \psi(x,y) = \psi(y,x) = \psi(y,y) = y$$

$$\pi(x) = y, \quad \pi(y) = x.$$

Then π is an involution, and $\mathscr{S}$ is selectively paired and strongly $\mathfrak{M}$-expansive. The strong $\mathfrak{M}$-expansiveness follows as a very special case of [1]. For the ordering defined by taking $x > y$ is a distributive lattice-ordering in which ϕ and ψ are the lattice operations. Now taking $f_j(x) = a_j$, $f_j(y) = 1$, (3.4) becomes the following inequality observed by Preston [13, Lemma 2]:

$$0 \leqq a_1 \leqq a_3,\ 0 \leqq a_2 \leqq a_3,\ 0 \leqq a_4,\ a_1 a_2 \leqq a_3 a_4 \Rightarrow a_1 + a_2 \leqq a_3 + a_4. \tag{3.5}$$

Conversely putting $a_1 = f_1(x)f_2(y)$, $a_2 = f_1(y)f_2(x)$, $a_3 = f_3(x)f_4(y)$, $a_4 = f_3(y)f_4(x)$ in (3.5) gives:

$$\mathbf{f} \in \mathfrak{R} \Rightarrow f_1(x)f_2(y) + f_1(y)f_2(x) \leqq f_3(x)f_4(y) + f_3(y)f_4(x) \tag{3.6}$$

and adding in the additional inequalities $f_1(x)f_2(x) \leqq f_3(x)f_4(x)$ and $f_1(y)f_2(y) \leqq f_3(y)f_4(y)$ gives:

$$\mathbf{f} \in \mathfrak{R} \Rightarrow (f_1(x) + f_1(y))(f_2(x) + f_2(y)) \leqq (f_3(x) + f_3(y))(f_4(x) + f_4(y)).$$

Thus the $\mathfrak{M}$-expansiveness of $\mathscr{S}$ is effectively equivalent to (3.5) and also to (3.6).

The only other selective pairings on X are (ψ, ϕ), (ϕ_0, ψ_0) and (ψ_0, ϕ_0), where $\phi_0(z, w) = z$ and $\psi_0(z, w) = w$. It is now easy to verify that (3.6) remains valid when compatibility refers to any of these pairs.

We turn next to arbitrary selectively paired spaces.

Proposition 3.4. *Any selectively paired measure space $\mathscr{S}$ is strongly $\mathfrak{M}$-expansive.*

Proof. Consider a μ-compatible function $\mathbf{f}$, and assume without loss that f_3 and f_4 are μ-integrable (Remark 3.1(a)). For any (x, y) in $X^2 \backslash \Delta$, let ϕ_{xy} and ψ_{xy} be the restrictions of ϕ and ψ to the two-point set $\{x, y\}$. These form a selective pairing on $\{x, y\}$, and we can ask whether the funciton $\mathbf{f}_{xy}$ obtained by restricting $\mathbf{f}$ to $\{x, y\}$ is compatible with (ϕ_{xy}, ψ_{xy}). To check this, it is required to verify that

$$f_1(x')f_2(y') \leqq f_3(\phi(x', y'))f_4(\psi(x', y')) \tag{3.7}$$

whenever $(x', y') = (x, x)$, (x, y), (y, x) or (y, y). Since $\mathbf{f}$ is μ-compatible, (3.7) is valid when $(x', y') = (x, y)$ or (y, x) for μ^2-almost all pairs (x, y) in $X^2 \backslash \Delta$. Also by Proposition 2.1, $\mathbf{f}$ is diagonally μ-compatible, so by (2.2), (3.7) holds for $(x', y') = (x, x)$ for μ-almost all x, and hence for μ^2-almost all pairs (x, y). Similarly (3.7) holds for $(x', y') = (y, y)$ for μ^2-almost all (x, y). Thus for μ^2-almost all (x, y) in $X^2 \backslash \Delta$, $\mathbf{f}_{xy}$ is compatible with the selective pairing (ϕ_{xy}, ψ_{xy}) on the two-point set $\{x, y\}$, so by (3.6)

$$f_1(x)f_2(y) + f_1(y)f_2(x) \leqq f_3(x)f_4(y) + f_3(y)f_4(x). \tag{3.8}$$

For (x, y) in Δ, (3.8) is equivalent to (2.1), so (3.8) holds μ^2-a.e. in Δ, and hence μ^2-a.e. in X^2. Integrating with respect to μ^2 over X^2 gives

$$2\mu(f_1)\mu(f_2) \leqq 2\mu(f_3)\mu(f_4).$$

There is a converse to Proposition 3.4.

Proposition 3.5. *Let X be any set, $\mathscr{F}_c$ be the σ-algebra of all countable and all cocountable subsets of X, (ϕ, ψ) be a mapping of X^2 into X^2, and suppose that $(X, \mathscr{F}_c, \mu, \phi, \psi)$ is expansive for every measure μ on $(X, \mathscr{F}_c)$. Then (ϕ, ψ) is selective.*

Proof. Take $E_1 = E_2 = \{x\}$ and $\mu = \delta_x$, where δ_x is the unit mass at x. Then Remark 3.1(b) gives

$$1 \leqq \delta_x\{\phi(x, x)\}\,\delta_x\{\psi(x, x)\}.$$

Hence $\phi(x, x) = \psi(x, x) = x$.

Now for distinct x and y take $E_1=\{x\}$, $E_2=\{y\}$, and $\mu=2\delta_x+\delta_y$. Then Remark 3.1(b) gives

$$2\leqq(2\delta_x\{\phi(x,y)\}+\delta_y\{\phi(x,y)\})(2\delta_x\{\psi(x,y)\}+\delta_y\{\psi(x,y)\}).$$

Hence $\delta_x\{\phi(x,y)\}$ and $\delta_x\{\psi(x,y)\}$ are not both zero, i.e. $\phi(x,y)=x$ or $\psi(x,y)=x$. Similarly $\phi(x,y)=y$ or $\psi(x,y)=y$.

We turn now to our main consideration of direct products. The following lemma is an abstract version of [2, Lemma 1].

Lemma 3.6. *Let $\mathscr{S}$ be the direct product of two paired measure spaces $\mathscr{S}_1$ and $\mathscr{S}_2$, and suppose that $\mathscr{S}_1$ is (strongly) $\mathfrak{M}$-expansive. Let $\mathbf{f}: X\to[0,\infty]^4$ be (μ-)compatible with (ϕ,ψ), and put*

$$g_j(x_2)=\int_{X_1} f_j(x_1,x_2)\,d\mu_1(x_1).$$

Then $\mathbf{g}$ is (μ_2-)compatible with (ϕ_2,ψ_2).

Proof. We consider the case when $\mathbf{f}$ is μ-compatible, the other being similar. For μ_2^2-almost all (x_2,y_2) in X_2^2, the function $\mathbf{h}: X_1\to[0,\infty]^4$ defined by (2.13) is μ_1-compatible with (ϕ_1,ψ_1), so, assuming that $\mathscr{S}_1$ is strongly $\mathfrak{M}$-expansive,

$$\mu_1(h_1)\,\mu_1(h_2)\leqq\mu_1(h_3)\,\mu_1(h_4).$$

This gives immediately

$$g_1(x_2)\,g_2(y_2)\leqq g_3(\phi_2(x_2,y_2))\,g_4(\psi_2(x_2,y_2)) \qquad \mu_2^2\text{-a.e. } (x_2,y_2).$$

The next result is the main theorem concerning direct products, and is a measure-theoretic analogue of [2, Theorem 1].

Theorem 3.7. *The direct product of any finite family of (strongly) $\mathfrak{M}$-expansive paired measure spaces is (strongly) $\mathfrak{M}$-expansive.*

Proof. By induction it suffices to consider the product of two spaces. Using the notation of Lemma 3.6, for any $\mathbf{f}$ (μ-)compatible with (ϕ,ψ), $\mathbf{g}$ is (μ_2-)compatible with (ϕ_2,ψ_2). Since $\mathscr{S}_2$ is (strongly) $\mathfrak{M}$-expansive,

$$\mu_2(g_1)\,\mu_2(g_2)\leqq\mu_2(g_3)\,\mu_2(g_4).$$

But, by Fubini's theorem, $\mu_2(g_j)=\mu(f_j)$, and the theorem follows.

As in [3, 8], it is possible to extend Theorem 3.7 to cover infinite products. However at this level of abstraction, there is an additional technical complication.

Theorem 3.8. *Let $\mathscr{S}$ be the direct product of a family $\{\mathscr{S}_\lambda: \lambda\in\Lambda\}$ of $\mathfrak{M}$-expansive paired probability spaces. Suppose that there is no $\mathscr{F}$-measurable set E such that $\mu(E)=1$ and $(E\times E)\cap\phi^{-1}(E)\cap\psi^{-1}(E)$ is empty. Then $\mathscr{S}$ is $\mathfrak{M}$-expansive. In particular if each $\mathscr{S}_\lambda$ is diagonally invariant and $\mathfrak{M}$-expansive, then $\mathscr{S}$ is $\mathfrak{M}$-expansive.*

Proof. Take a function $\mathbf{f}: X \to [0,\infty]^4$ which is compatible with (ϕ,ψ). Since $\mathbf{f}$ is $\mathscr{F}$-measurable, there is a countable subset Λ_0 of Λ such that $f(x)=f(y)$ whenever $x_\lambda = y_\lambda$ for all λ in Λ_0. Thus replacing Λ by Λ_0, we may assume that Λ is countable, and hence that $\Lambda = \mathbf{N}$.

Let $\mathscr{T}_m = (Z_m, \mathscr{G}_m, \nu_m, \rho_m, \sigma_m)$ and $\mathscr{T}'_m = (Z'_m, \mathscr{G}'_m, \nu'_m, \rho'_m, \sigma'_m)$ be the respective direct products of $\{\mathscr{S}_n : n>m\}$ and $\{\mathscr{S}_n : 1 \leqq n \leqq m\}$, and identify $\mathscr{S}$ with $\mathscr{T}'_m \times \mathscr{T}_m$. By Theorem 3.7, $\mathscr{T}'_m$ is $\mathfrak{M}$-expansive. Thus if

$$\tilde{f}_{mj}(z'_m, z_m) = f_{mj}(z_m) = \int_{Z'_m} f_j(w'_m, z_m)\, d\nu'_m(w'_m) \quad (z'_m \in Z'_m, z_m \in Z_m)$$

Lemma 3.6 shows that $\mathbf{f}_m$ is compatible with (ρ_m, σ_m), i.e.

$$f_{m1}(z_m) f_{m2}(w_m) \leqq f_{m3}(\rho_m(z_m, w_m)) f_{m4}(\sigma_m(z_m, w_m)) \quad (z_m, w_m \in Z_m)$$

or equivalently,

$$\tilde{f}_{m1}(x) \tilde{f}_{m2}(y) \leqq \tilde{f}_{m3}(\phi(x,y)) \tilde{f}_{m4}(\psi(x,y)) \quad (x, y \in X). \tag{3.9}$$

By the reverse version of the Fubini-Jessen theorem [7, Theorem III.11.27], $\tilde{f}_{mj}(x)$ converges to the constant $\mu(f_j)$ as $m \to \infty$ for all x in some set E with $\mu(E) = 1$. Taking (x, y) in the non-empty set $(E \times E) \cap \phi^{-1}(E) \cap \psi^{-1}(E)$ and letting $m \to \infty$ in (3.9) gives (3.2).

If each $\mathscr{S}_\lambda$ and hence $\mathscr{S}$ is diagonally invariant, choosing x in E and putting $y = x$ in (3.9) gives the result.

Corollary 3.9. *The direct product of any family of selectively paired probability spaces is $\mathfrak{M}$-expansive.*

Proof. This is immediate from Proposition 3.4, Theorem 3.8 and the fact that a selective pairing is diagonally invariant.

In the notation of Theorem 3.8, Remark 3.1(b) shows that $\mu^*_\lambda(\phi_\lambda(X^2_\lambda)) = \mu^*_\lambda(\psi_\lambda(X^2_\lambda)) = 1$, so $\mu^*(\phi(X^2)) = \mu^*(\psi(X^2)) = 1$. Hence if $\mu(E) > 0$, then $\phi^{-1}(E)$ and $\psi^{-1}(E)$ are non-empty. However it is not clear that $(E \times E) \cap \phi^{-1}(E) \cap \psi^{-1}(E)$ is non-empty, even if $\mu(E) = 1$, nor indeed whether Theorem 3.8 remains valid if this condition is dropped. Note however that if $(E \times E) \cap \phi^{-1}(E) \cap \psi^{-1}(E)$ is empty, then $(2\chi_E, \chi_E, 2-\chi_E, 2-\chi_E)$ is compatible. Hence if $\mathscr{S}$ is $\mathfrak{M}$-expansive, then $\mu(E) < 1$.

Strong $\mathfrak{M}$-expansiveness is rarely preserved under infinite direct products. For instance the μ-compatible function $(1, 1, 1-\chi_E, 1)$ as constructed in Example 2.3 does not satisfy (3.2). Indeed using Proposition 2.2 and the forward version of the Fubini-Jessen theorem, it is possible to show that a product $\mathscr{S}$ of diagonally settled, diagonally invariant, strongly $\mathfrak{M}$-expansive, measurably paired spaces is strongly $\mathfrak{M}$-expansive if and only if $\mathscr{S}$ is diagonally settled. A simple modification of the proof of Theorem 3.8 shows that this does occur if (and only if) there is no $\mathscr{F}$-measurable set E with $\mu(E) = 1$ such that $\phi^{-1}(E) \cap \psi^{-1}(E)$ is μ^2-null.

Example 3.10. Suppose that X is a lattice and v_1 and v_2 are functions of X into $[0,\infty]$ satisfying (1.9), viz. $v_1(x) \leqq v_2(y)$ whenever $x \leqq y$. Then $(1, v_1, v_2, 1)$ is

compatible with $(\vee, \wedge)$. Furthermore if $\mathbf{f}$ is compatible with $(\vee, \wedge)$, then by (2.6), so is $(f_1, v_1 f_2, v_2 f_3, f_4)$.

Now if $(X, \mathscr{F}, \mu)$ is a product of totally ordered measure spaces and X is given the product ordering, then X is a lattice, and $\mathscr{S} = (X, \mathscr{F}, \mu, \vee, \wedge)$ is $\mathfrak{M}$-expansive by Corollary 3.9. Thus for v_1, v_2 and $\mathbf{f}$ as above we recover the inequality (1.8), viz.

$$\mu(f_1)\mu(v_1 f_2) \leqq \mu(v_2 f_3)\mu(f_4).$$

Thus in the terminology of [11], $\mathscr{S}$ is an "FKG-space". The inequality (1.8) was first proved in [13] for finite products and in [3] for infinite products under the additional assumptions that $v_1 = v_2$, $f_1 = f_3$ and $f_2 = f_4$. Edwards [8] and Kemperman [11] gave simplified proofs and allowed v_1 and v_2 to differ, but they still required that $f_1 = f_3$ and $f_2 = f_4$. If the proof of Corollary 3.9 is followed through, it seems that our proof of the more general result is simpler than those in [8, 11] both in finite cases and in the infinite extension. Furthermore for a finite product, Proposition 3.4 and Theorem 3.7 show that $\mathscr{S}$ is strongly $\mathfrak{M}$-expansive, so our result also improves that of [13] in that for (1.8) to be valid it is sufficient that (1.7) holds μ^2-a.e. (and that $v_1(x) \leqq v_2(x \vee y)\,\mu^2$-a.e.). However to establish this required considerably more technical detail in the proof of Proposition 3.4.

Allowing for example f_1 and f_3 to differ makes it pointless to seek measures on $\{(x, y) \in X^2 : x \leqq y\}$ with certain marginals as was successfully done in [13].

4. Explosiveness

Again following [2], we now introduce the concept of explosiveness for a paired measure space $\mathscr{S}$. An $\mathscr{F}$-measurable function $\mathbf{g}: X \to [0, \infty]^8$ is *bicompatible* (resp. *μ-bicompatible*) with (ϕ, ψ) if:

$$g_1(x)\, g_2(x')\, g_3(y)\, g_4(y') \leqq g_5(\phi(x, y'))\, g_6(\psi(y, x'))\, g_7(\phi(y, x'))\, g_8(\psi(x, y')) \quad (4.1)$$

for all (resp. μ^4-almost all) (x, x', y, y') in X^4. For subsets F_1 and F_2 of X^2, put

$$\phi\psi(F_1, F_2) = \{(\phi(x, y'), \psi(y, x')) : (x, x') \in F_1, (y, y') \in F_2\}. \quad (4.2)$$

A 4-tuple (F_1, F_2, F_3, F_4) of $\mathscr{F}^2$-measurable sets is *bicompatible* with (ϕ, ψ) if

$$\phi\psi(F_1, F_2) \subset F_3 \quad \text{and} \quad \phi\psi(F_2, F_1) \subset F_4. \quad (4.3)$$

The space $\mathscr{S}$ is *explosive* if

$$\mu^2(F_1)\,\mu^2(F_2) \leqq \mu^2(F_3)\,\mu^2(F_4)$$

for all (F_1, F_2, F_3, F_4) bicompatible with (ϕ, ψ); *(strongly) $\mathfrak{M}$-explosive* if

$$\left(\int_{F_1} g_1 \otimes g_2 \, d\mu^2\right)\left(\int_{F_2} g_3 \otimes g_4 \, d\mu^2\right) \leqq \left(\int_{F_3} g_5 \otimes g_6 \, d\mu^2\right)\left(\int_{F_4} g_7 \otimes g_8 \, d\mu^2\right) \quad (4.4)$$

for all (μ-)bicompatible $\mathbf{g}$ and bicompatible (F_1, F_2, F_3, F_4), where

$$(h \otimes h')(x, x') = h(x)\, h'(x') \quad (x, x' \in X).$$

For counting measure on a finite set, these definitions of explosiveness and $\mathfrak{M}$-explosiveness reduce to those of [2].

Define a pairing $(\overline{\phi\psi}, \underline{\phi\psi})$ on X^2 by:

$$\overline{\phi\psi}((x,x'),(y,y')) = (\phi(x,y'),\psi(y,x'))$$
$$\underline{\phi\psi}((x,x'),(y,y')) = (\phi(y,x'),\psi(x,y'))$$

so that $\underline{\phi\psi}(z) = \overline{\phi\psi}(z^*) = \overline{\psi\phi}(z)^*$. (The asterisks here refer to the operation changing the order of pairs in $X^2 \times X^2$. In [2], $\overline{\phi\psi}$ and $\underline{\phi\psi}$ were denoted by ϕ^2 and ψ^2 respectively, but we feel such notation might be in conflict with that adopted elsewhere in this paper.) Then it is clear from (2.1), (2.3), (4.1), (4.2) and (4.3) that

$$\mathbf{g} \text{ is } (\mu\text{-})\text{bicompatible with } (\phi,\psi) \Leftrightarrow$$

$$(g_1 \otimes g_2, g_3 \otimes g_4, g_5 \otimes g_6, g_7 \otimes g_8) \text{ is } (\mu\text{-})\text{compatible with } (\overline{\phi\psi}, \underline{\phi\psi}) \tag{4.5}$$

$$\phi\psi(F_1,F_2) = \overline{\phi\psi}(F_1 \times F_2), \qquad \phi\psi(F_2,F_1) = \underline{\phi\psi}(F_1 \times F_2)$$

$$(F_1,F_2,F_3,F_4) \text{ is bicompatible with } (\phi,\psi) \Leftrightarrow$$

$$(F_1,F_2,F_3,F_4) \text{ is compatible with } (\overline{\phi\psi}, \underline{\phi\psi}). \tag{4.6}$$

The following is a measure-theoretic extension of [2, Theorem 6].

Theorem 4.1. *The following are equivalent for a paired measure space* $\mathscr{S}$:

(i) $(X^2, \mathscr{F}^2, \mu^2, \phi \times \psi^*, \psi \times \phi^*)$ is *(strongly)* $\mathfrak{M}$*-expansive*

(ii) $(X^2, \mathscr{F}^2, \mu^2, \overline{\phi\psi}, \underline{\phi\psi})$ *is (strongly)* $\mathfrak{M}$*-expansive*

(iii) $\mathscr{S}$ *is (strongly)* $\mathfrak{M}$*-explosive*

(iii)′ *For any* $\mathbf{g}$ $(\mu$*-)bicompatible with* (ϕ,ψ),

$$\mu(g_1)\mu(g_2)\mu(g_3)\mu(g_4) \leqq \mu(g_5)\mu(g_6)\mu(g_7)\mu(g_8)$$

(iv) $\mathscr{S}$ *is (strongly)* $\mathfrak{M}$*-expansive*

(v) $(X, \mathscr{F}, \mu, \psi^*, \phi^*)$ *is (strongly)* $\mathfrak{M}$*-expansive.*

Proof. We shall consider the weaker properties, the equivalence of the stronger versions being analogous.

(i) $\Leftrightarrow$ (ii). It is readily verified (cf. [2, p. 285]) that (f_1,f_2,f_3,f_4) is compatible with $(\overline{\phi\psi}, \underline{\phi\psi})$ if and only if (f_1,f_2^*,f_3,f_4^*) is compatible with $(\phi \times \psi^*, \psi \times \phi^*)$. Furthermore

$$\mu^2(f_1)\mu^2(f_2) = \mu^2(f_1)\mu^2(f_2^*), \qquad \mu^2(f_3)\mu^2(f_4) = \mu^2(f_3)\mu^2(f_4^*).$$

(ii) $\Rightarrow$ (iii). The inequality (4.4) follows from (ii) via (4.5), (4.6) and (3.3).

(iii) $\Rightarrow$ (iii)′. This is immediate on restricting (4.4) to the case $F_j = X^2$.

(iii)′ $\Rightarrow$ (iv). For $\mathbf{f}$ compatible with (ϕ,ψ), put

$$g_1 = g_3 = f_1, \quad g_2 = g_4 = f_2, \quad g_5 = g_7 = f_3, \quad g_6 = g_8 = f_4.$$

Then $\mathbf{g}$ is bicompatible with (ϕ,ψ), so

$$\mu(f_1)^2\mu(f_2)^2 = \mu(g_1)\mu(g_2)\mu(g_3)\mu(g_4) \leqq \mu(g_5)\mu(g_6)\mu(g_7)\mu(g_8) = \mu(f_3)^2\mu(f_4)^2.$$

(iv) $\Leftrightarrow$ (v). By (2.4) and (2.5), (f_1, f_2, f_3, f_4) is compatible with (ϕ, ψ) if and only if (f_2, f_1, f_4, f_3) is compatible with (ψ^*, ϕ^*).

(iv) $\Rightarrow$ (i). This follows immediately from the already proved equivalence of (iv) with (v) together with Theorem 3.7.

Proposition 4.2. *Consider the following properties of a paired measure space* $\mathscr{S}$:

(i) $(X^2, \mathscr{F}^2, \mu^2, \phi \times \psi^*, \psi \times \phi^*)$ *is expansive*

(ii) $(X^2, \mathscr{F}^2, \mu^2, \overline{\phi\,\psi}, \underline{\phi\,\psi})$ *is expansive*

(iii) $\mathscr{S}$ *is explosive*

(iv) $\mathscr{S}$ *is expansive*

(v) $(X, \mathscr{F}, \mu, \psi^*, \phi^*)$ *is expansive.*

The following implications are valid:

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)} \Leftrightarrow \text{(v)}.$$

Proof. Specialising the proof of Theorem 4.1 to the case of characteristic functions leads to a proof of all implications except (iii) $\Rightarrow$ (ii), which is immediate from (4.6).

5. Finite Distributive Lattices

Example 3.3 shows that $(\{0,1\}, 2^{\{0,1\}}, \mu_c, \vee, \wedge)$ is an $\mathfrak{M}$-expansive space. Applying Theorem 3.7 to a finite family of copies of this space shows that for any finite set Λ, $\mathscr{S}(\Lambda) = (2^\Lambda, 2^{2^\Lambda}, \mu_c, \cup, \cap)$ is also $\mathfrak{M}$-expansive. Any finite distributive lattice L can be embedded in 2^Λ for some finite set Λ [4, Corollary, p. 59], so it follows from Remark 3.1(c) that $\mathscr{S}_L = (L, 2^L, \mu_c, \vee, \wedge)$ is $\mathfrak{M}$-expansive, i.e. inequality (1.10) holds. (An extension to infinite lattices can then be made by a simple limiting argument.) This result was first obtained in [1], and itself has many important consequences as listed in [2, Sect. 9]. The expansiveness of $\mathscr{S}_L$ gives inequality (1.2).

Conversely let L be a finite lattice such that $\mathscr{S}_L$ is $\mathfrak{M}$-expansive. Then $\mathscr{S}_L$ is an FKG-space (see Example 3.10) and hence L is distributive [11, Theorem 7]. Thus $\mathscr{S}_L$ is $\mathfrak{M}$-expansive if and only if $\mathscr{S}_L$ is an FKG-space, or, equivalently, L is distributive.

For any finite set S, there is an involution π on $\mathscr{S}(S)$ given by $\pi(A) = S \setminus A$. Let F_1 and F_2 be functions of 2^S into 2^L, and put

$$\begin{aligned} F_3(C) &= \bigcup_{A \cup B = C} F_1(A) \vee F_2(B) \\ F_4(C) &= \bigcup_{A \cap B = C} F_1(A) \wedge F_2(B) \\ f_j(A) &= |F_j(A)| \quad (j = 1,2,3,4). \end{aligned} \tag{5.1}$$

Then (1.2) shows that $\mathbf{f}$ is compatible with $(\cup, \cap)$ on 2^S. Now applying Proposition 3.2 gives the inequality

$$\sum_{A \subset S} |F_1(A)|\,|F_2(S \setminus A)| \leqq \sum_{C \subset S} |F_3(C)|\,|F_4(S \setminus C)| \tag{5.2}$$

which (apart from notational changes) is the same as (1.3). Thus [6, Theorem 1] can be deduced from (1.2) and $\mathfrak{M}$-expansiveness. Since (1.2) is itself a consequence of expansiveness, (5.2) could be obtained directly from a single application of expansiveness. Rather than giving explicit details of this, we prefer to show how (5.2) can be obtained more fundamentally from an application of (1.2) in a different lattice.

Let $\mathscr{L}$ be the lattice $2^S \times L \times \tilde{L}$ with the product ordering, where $\tilde{L}$ is the lattice obtained from L by reversing the ordering in L. Put

$$\mathscr{A} = \{(A, x_1, x_2) \in \mathscr{L} : x_1 \in F_1(A), x_2 \in F_2(S \setminus A)\}$$
$$\mathscr{B} = \{(B, y_1, y_2) \in \mathscr{L} : y_1 \in F_2(B), y_2 \in F_1(S \setminus B)\}.$$

Then

$$\mathscr{A} \vee \mathscr{B} \subset \{(C, z_1, z_2) \in \mathscr{L} : z_1 \in F_3(C), z_2 \in F_4(S \setminus C)\}$$
$$\mathscr{A} \wedge \mathscr{B} \subset \{(D, w_1, w_2) \in \mathscr{L} : w_1 \in F_4(D), w_2 \in F_3(S \setminus D)\}$$
$$|\mathscr{A}| = |\mathscr{B}| = \sum_{A \subset S} |F_1(A)|\,|F_2(S \setminus A)|$$
$$|\mathscr{A} \vee \mathscr{B}|, \quad |\mathscr{A} \wedge \mathscr{B}| \leqq \sum_{C \subset S} |F_3(C)|\,|F_4(S \setminus C)|.$$

Thus (1.2) applied to $\mathscr{A}$ and $\mathscr{B}$ gives (the square of) (5.1) immediately.

Recall that a subset U of L is an *up-set* if

$$x \in U, \quad y \in L, \quad x \leqq y \Rightarrow y \in U.$$

Down-sets are defined similarly, and a down-set of the form $\{y \in L : y \leqq x\}$ is an *ideal* in L.

Daykin [6] has obtained a number of other inequalities concerning up-sets, down-sets and ideals in 2^A, but it was not clear from his methods that the results could be transferred to arbitrary finite distributive lattices. (An up-set in one lattice may fail to be an up-set in a larger lattice.) We shall now show that they do remain valid by deducing them from (5.2).

Theorem 5.1. *Let I and J be ideals in a finite distributive lattice L, S be a finite set, U, V, W and X be increasing functions from 2^S into the set of up-sets in L (ordered by inclusion), and D and E be decreasing functions of 2^S into the set of down-sets in L. Suppose that $I \subset J$, $U(A) \subset V(A)$, $W(A) \subset X(A)$ and $D(A) \subset E(A)$ for all A in 2^S. Then*

$$|I| \sum_{A \subset S} |J \cap V(A) \cap D(A)| + |J| \sum_{A \subset S} |I \cap U(A) \cap E(A)|$$
$$\leqq \sum_{C \subset S} |I \cap U(C)|\,|J \cap D(S \setminus C)| + \sum_{C \subset S} |I \cap E(C)|\,|J \cap V(S \setminus C)| \tag{5.3}$$

$$\sum_{A \subset S} |I \cap U(A)|\,|J \cap X(S \setminus A)| + \sum_{A \subset S} |I \cap W(A)|\,|J \cap V(S \setminus A)|$$
$$\leqq |I| \sum_{C \subset S} |J \cap X(C) \cap V(C)| + |J| \sum_{C \subset S} |I \cap U(C) \cap W(C)|. \tag{5.4}$$

Proof. Let S' be the set obtained by adjoining one additional point ω to S, and for A contained in S, let $A'=A\cup\{\omega\}$. Put

$$F_1(A)=I\cap U(A)\cap E(A) \qquad F_1(A')=J\cap V(A)\cap D(A)$$
$$F_2(A)=I \qquad F_2(A')=J.$$

Then F_1 and F_2 are functions from $2^{S'}$ into 2^L, and if F_3 and F_4 are defined by (5.1) with S' replacing S,

$$F_3(C)=\bigcup_{A\cup B=C}(I\cap U(A)\cap E(A))\vee I$$
$$\subset I\cap U(C)$$
$$F_3(C')=\bigcup_{A\cup B=C}\{[(I\cap U(A)\cap E(A))\vee J]\cup[(J\cap V(A)\cap D(A))\vee I]$$
$$\cup[(J\cap V(A)\cap D(A))\vee J]\}$$
$$\subset J\cap V(C).$$

Similarly

$$F_4(C)\subset I\cap E(C), \qquad F_4(C')\subset J\cap D(C).$$

Now applying (5.2) with S replaced by S' gives

$$\begin{aligned}
&|I|\sum_{A\subset S}|J\cap V(A)\cap D(A)|+|J|\sum_{A\subset S}|I\cap U(A)\cap E(A)|\\
&=\sum_{A\subset S}|F_1(A)|\,|F_2(S'\setminus A')|+\sum_{A\subset S}|F_1(A)|\,|F_2(S'\setminus A)|\\
&=\sum_{B\subset S'}|F_1(B)|\,|F_2(S'\setminus B)|\\
&\leqq\sum_{B\subset S'}|F_3(B)|\,|F_4(S'\setminus B)|\\
&\leqq\sum_{C\subset S}|I\cap U(C)|\,|J\cap D(S\setminus C)|+\sum_{C\subset S}|I\cap E(C)|\,|J\cap V(S\setminus C)|.
\end{aligned}$$

A similar argument, the details of which are left to the reader, leads to (5.4).

The inequalities (5.3) and (5.4) were obtained in [6, Theorems 2, 3] in the case when $L=2^A$ for some finite set A. Combining the methods of this section, they could each have been deduced from a single application of expansiveness in $\mathscr{L}_{\mathscr{L}'}$, where $\mathscr{L}'=2^{S'}\times L\times\tilde{L}$. Putting $S=\emptyset$, $I=J=L$, $U(S)=\emptyset$, $V(S)=U$, $D(S)=E(S)=D$ gives (1.1) as a special case of (5.3).

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On Local Times for Non-homogeneous Markov Processes

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Summary. Random sets of the type $M_B(\omega)=\{t\colon (t, x_t(\omega))\in B\}$ associated with measurable sets B from the phase space of a non-homogeneous Markov process are considered.

The process is supposed to satisfy Dynkin's regularity conditions. When the set B coincides with the set of all points that are regular for it, special properties (well known in the homogeneous case) appear: namely, $\overline{M}_B$ is a.s. perfect and there exist predictable (under certain conditions also continuous) local times for B.

§ 1. Introduction

1.1. In this section we introduce the class of processes that shall concern us throughout the paper. These are essentially Dynkin's "regular Markov processes" [4], subject to the restriction that the time parameter ranges over $[0, \infty)$ and not $(-\infty, +\infty)$. This restriction is imposed by some results from martingale theory involved in the proof of the representation theorem of excessive functions as potentials of additive functionals. Other slight modifications also occurring we start by summarizing our hypotheses.

Assume we are given:

a) for any $t\in R_+$ the borelian space $(E_t, \mathscr{E}_t)$ and two distinct points a_t, b_t that do not belong to E_t. The space $E=\{(t,x)\colon t\in R_+, x\in E_t\}$ is called the phase space.

b) a normal transition function $p(t,x;u,\Gamma)$, $t<u$, $x\in E_t$, $\Gamma\in\mathscr{E}_u$ extended to E by setting $p(t,x;u,\Gamma)=0$ for $t\geqq u$. We denote by $\mathscr{E}$ the σ-algebra generated on E by the family of functions $\{p(t,x;u,\Gamma); u\in R_+, \Gamma\in\mathscr{E}_u\}$.

c) a family of mappings $\{x_t(\omega)\}_t$ defined on a space Ω such that for some α and β also defined on Ω, $x_t(\omega)\in E_t$ for $\alpha(\omega)\leqq t<\beta(\omega)$; $x_t(\omega)=a_t$, $t<\alpha(\omega)$; $x_t(\omega)=b_t$, $t\geqq\beta(\omega)$. The variables α and β are respectively called birth and death moment for the process.

We define on Ω the σ-algebras generated by the process, namely for any $s<t$, $\mathscr{F}^0_{st}$ is the σ-algebra generated by the events $\{x_u\in\Gamma\}$, $s\leqq u\leqq t$, $\Gamma\in\mathscr{E}_u$; $\mathscr{F}^0_t=\mathscr{F}^0_{0t}$; $\mathscr{F}^0=\mathscr{F}^0_{0\infty}$.

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Finally, assume that we are given

d) a family of probabilities $\mathscr{K}$ on $(\Omega, \mathscr{F}^0)$ with the property that for any $(t,x)\in E$ there exists $P_{tx}\in\mathscr{K}$ such that $P_{tx}(\alpha=t)=1$ and $P_{tx}\{x_u\in\Gamma\}=p(t,x;u,\Gamma)$ for any $u>t$, $\Gamma\in\mathscr{E}_u$.

The family $\{x_t,\mathscr{F}_t^0,\mathscr{K}\}$ is said to have the *Markov property* if for any $P\in\mathscr{K}$, $u>t$, $\Gamma\in\mathscr{E}_u$ we have

(M) $P\{x_u\in\Gamma|\mathscr{F}_t^0\}=p(t,x_t;u,\Gamma)$, P-a.s. on $t\in[\alpha,\beta)$.

The Markov family $\{x_t,\mathscr{F}_t^0,\mathscr{K}\}$ is called *regular* if

(R) $\lim_{t\downarrow s} p(t,x_t;u,\Gamma)=p(s,x_s;u,\Gamma)$, a.s. on $[\alpha,\beta)$.

Two other additional conditions (appearing in some results from [4]) are also imposed upon $\{x_t,\mathscr{F}_t^0,\mathscr{K}\}$, namely:

(A.1) the transition function p separates the states

(A.2) for any $(t,x)\in E$, $P_{tx}(x_t=x)=1$.

1.2. In the sequel the basic σ-algebras will be the standard completion of $\mathscr{F}_{st}^0$, therefore we shall insist on some of their properties.

Let $\mathscr{F}^P$ be the completion of $\mathscr{F}^0$ with respect to $P\in\mathscr{K}$ and $\mathscr{F}=\bigcap_{P\in\mathscr{K}}\mathscr{F}^P$; $\mathscr{F}_{st}^P$ will be the σ-algebra generated by $\mathscr{F}_{st}^0$ and all sets of P-measure zero in $\mathscr{F}^P$; $\mathscr{F}_{st}=\bigcap_{P\in\mathscr{K}}\mathscr{F}_{st}^P$.

In order to show that for any $P\in\mathscr{K}$ the family $\{\mathscr{F}_t^P\}_t$ is right continuous we have only to notice that due to the condition (R) it is possible to write the relation in (M) with $\mathscr{F}_{t+}^P$ replacing $\mathscr{F}_t^P$ and then to apply an argument similar to the one in [8], Theorem 13.

We recall from [4] that whenever (A.1) is fulfilled, for any t fixed, $\mathscr{E}_t$ appears to be generated by the family of functions $\{p(t,x;u,f);\ u\in T, f\in\mathscr{W}_u\}$, where T is any countable set dense in R_+ and $\mathscr{W}_u$ a countable family of measure determining functions in $\mathscr{E}_u$. This fact and another appeal to (R) ensures that up to completion it does not count whether initially we considered the events $\{x_u\in\Gamma\}$, $s\leqq u\leqq t$ or $\{x_u\in\Gamma\}$, $s<u\leqq t$. This is valid for $\mathscr{F}^t$, the σ-algebra obtained from $\mathscr{F}_{t\infty}^0$ by standard completion, which will play an essential role when stating the strong Markov property.

In the sequel our "regular" Markov process will be $\{x_t,\mathscr{F}_t,\mathscr{K}\}$.

1.3. On E another σ-algebra denoted by $\mathscr{E}^{(n)}$ is introduced as follows: $f\in\mathscr{E}^{(n)}$ provided for any $P\in\mathscr{K}$ there exist f_1^P, $f_2^P\in\mathscr{E}$ such that $f_1^P\leqq f\leqq f_2^P$; $f_1^P(t,x_t)=f_2^P(t,x_t)$ for any $t\in[\alpha,\beta)$, P-a.s. (This is very similar to the σ-algebra of nearly borelian sets and therefore we maintained the notation, although initially there is no borelian structure on E.)

We are now in a position to formulate our main problem. With each $B\in\mathscr{E}^{(n)}$ associate the random set

$$M_B=\{(t,\omega):(t,x_t(\omega))\in B\}. \tag{1.1}$$

According to [4] this is a $\mathscr{K}$-optional[1] set with which the right continuous process

$$T_t^B(\omega)=\inf\{u>t:\ u\in M_B(\omega)\}$$

is in turn associated. We denote $T^B=T_\alpha^B$. For each t, T_t^B is a stopping time with respect to $\{\mathscr{F}_t\}_t$ and $T_t^B\in\mathscr{F}^t$.

We further denote

$$B^r=\{(t,x):\ P_{tx}(T^B=t)=1\}$$

$$f_B^\lambda(t,x)=P_{tx}\{\exp[-\lambda(T^B-t)]\},\qquad \lambda\in(0,\infty)^2.$$

We would like to find out a finite additive functional whose set of increase points coincides with $\overline{M}_B$ (the local time for B). Perhaps it would be more natural (taking into account the usual significance of the local time) to consider sets from $E'=\bigcup_t E_t$. But defining on E' the σ-algebra $\mathscr{G}$ given by those C for which $\{(t,x)\in E, x\in C\}\in\mathscr{E}$, and then considering random sets of the type $\{(t,\omega):\ x_t(\omega)\in C\}$, with $C\in\mathscr{G}$, this amounts to consider random sets of the type (1.1).

It is perhaps worthwhile to single out the two extreme situations that may occur. First, when all spaces coincide (i.e. $E_t=E'$, $\mathscr{E}_t=\mathscr{E}'$ for all $t\in R_+$) a very workable condition is that the mapping $(s,x)\to p(s,x;u,\Gamma)$ be measurable with respect to $\mathscr{B}(R_+)\otimes\mathscr{E}'$, which under (A.1) implies $\mathscr{B}(R_+)\otimes\mathscr{E}'=\mathscr{E}$ and everything is on a familiar ground. Secondly, when all spaces are disjoint (such processes are the "regularizations" obtained in [4, 6]) exactly the sets of type (1.1) are worth investigation. Anyway it is convenient to transfer everything on the phase space where one has the proper apparatus.

A key role will play an appropriate form of the strong Markov property (§ 2), from which general properties regarding the structure of $\overline{M}_B$ and other associated random sets, follow almost at once (§ 3). In § 4 is presented the main result, namely the existence of $\mathscr{K}$-predictable local times for those $B\in\mathscr{E}^{(n)}$ for which $B=B^r$.

§ 2. Preliminary Results

2.1. Strong Markov Property

Let Q_0 be the family of all real-valued processes $(\eta_t)_{t\in R_+}$ defined on Ω, having a.s. right continuous paths and such that for each t, η_t is $\mathscr{F}^t$-measurable.

Let Q be the family of all processes measurable with respect to the σ-algebra generated by Q_0 on $R_+\times\Omega$.

Theorem 2.1. *If η is a positive process from Q and τ is a stopping time, then denoting $F_\eta(t,x)=P_{tx}(\eta_t)$ we have for any $P\in\mathscr{K}$*

(SM) $P\{\eta_\tau|\mathscr{F}_\tau\}=F_\eta(\tau,x_\tau)$, P-a.s. on $\tau\in[\alpha,\beta)$.

[1] A process is said to be P-optional (P-predictable) if it is P-indistinguishable from an optional (predictable) one over $\{\Omega,\mathscr{F},\mathscr{F}_t,P\}$. It is $\mathscr{K}$-optional ($\mathscr{K}$-predictable) if this holds for any $P\in\mathscr{K}$

[2] The expectation with respect to P of the P-integrable random variable ξ is denoted by $P\xi$

The proof of this result is given in [6] for non-random life interval. The extension to the present situation is almost immediate. In the sequel whenever applying the strong Markov property we will in fact refer to this theorem. As an immediate but very important application we get

Lemma 2.1. *For any* $B\in\mathscr{E}^{(n)}$ *and any stopping time* σ *we have* $(T^B_\sigma, x_{T^B_\sigma})\in B\cup B^r$ *a.s. on* $\{T^B_\sigma<\beta\}$.

2.2. *Representation of Excessive Functions as Potentials of Additive Functionals* ([2])

Let $f(t,x)$ be a λ-excessive function. For each $P\in\mathscr{K}$ we consider the process

$$Z^P_u=\begin{cases}P\{e^{-\lambda\alpha}f(\alpha,x_\alpha)\,|\,\mathscr{F}_u\}, & u<\alpha\\ e^{-\lambda u}f(u,x_u), & u\in[\alpha,\beta)\\ 0, & u\geqq\beta\end{cases}$$

which is a supermartingale, a.s. right continuous on R_+.

The λ-excessive function is said to be a *λ-potential* if for any $P\in\mathscr{K}$ the supermartingale $\{Z^P_u,\mathscr{F}_u,P\}$ is uniformly integrable and $\lim\limits_{t\to\infty}Z^P_t=0$, P-a.s. The notions of *λ-potential of class* (D) and *regular λ-potential* are similarly defined.

The additive functionals considered in this paper are particular cases of those in [5], namely they are processes $(A_t)_t$ with a.s. increasing and right continuous paths; $A_\alpha=0$ a.s. and $A_t=A_\beta$ for all $t>\beta$ a.s.; for any $s<t$, $A_t-A_s\in\mathscr{F}_{st}$.

For any $\lambda\geqq0$ denote $u^\lambda_A(t,x)=P_{tx}\left\{\int_t^\beta e^{-\lambda(u-t)}dA_u\right\}$.

Theorem 2.2. *Let f be an excessive function which is a λ-potential of class* (D). *Then there exists a unique $\mathscr{K}$-predictable additive functional $(A_t)_t$ such that $u^\lambda_A(t,x)=f(t,x)$ for all $(t,x)\in E$.*

This additive functional is a.s. continuous if and only if f is a regular λ-potential.

§ 3. Elementary Properties of Markov Random Sets

These are rather straightforward generalizations of the ones presented in [9]. The fact that they hold in the non-homogeneous case shows that it is not the homogeneity that plays an essential role but the underlying strong Markov property.

For $B\in\mathscr{E}^{(n)}$ we consider M_B as in § 1, as well as the random sets associated with it: $\overline{M}_B$ the closure of M_B, $\overline{M}^d_B$ the right closure of M_B, $\overline{M}'_B$ the set of non isolated points of $\overline{M}_B$ and $\overline{M}'^d_B$ the set of all points in $\overline{M}'_B$ that are not isolated to the right.

In the sequel the notation $M_1\dot{\subseteq}M_2$, for M_1, M_2 random sets, means that $M_2\setminus M_1$ is P-evanescent for each $P\in\mathscr{K}$.

Lemma 3.1.

(i) $\overline{M}'^{d}_{B} \dot{\subseteq} \overline{M}_{B^r} \dot{\subseteq} \overline{M}'_{B}$

(ii) $\overline{M}_{B} \doteq \overline{M}_{B \cup B^r}$.

The sets $\overline{M}'_B$, $\overline{M}_B$ are optional and M_{B^r}, $M_{B\cup B^r}$ are $\mathscr{K}$-optional. With this in mind the proof goes exactly along the lines of the proof of Proposition 1 in [9] (with the proper replacement of the strong Markov property) and therefore will be omitted. As an illustration we prefer to give the proof of the following result.

Lemma 3.2. *Let N_B be the perfect kernel of $\overline{M}_B$ and let*

$$U_t(\omega) = \inf\{u > t;\ u \in N_B(\omega)\}, \qquad U = U_\alpha,$$
$$K = \{(t,x):\ P_{tx}(U = t) = 1\}.$$

We have $\overline{M}_k \doteq N_B$; $K^r = K$; for each $P \in \mathscr{K}$ the set $M_{B^r \setminus K}$ is P-indistinguishable from a countable union of graphs of stopping times.

Proof. We obviously have $\overline{M}_K = \overline{\bigcup_{r\in Q_+} [T_r^K]}$ and $N_B = \overline{\bigcup_{r\in Q_+} [U_r]}$ (Q_+ denotes the positive rationals). Since M_K is a.s. right closed we have $(T_r^K, x_{T_r^K}) \in K$ a.s. on $\{T_r^K < \beta\}$, which by the strong Markov property implies $U_{T_r^K} = T_r^K$ a.s. on $\{T_r^K < \beta\}$ and hence $\overline{M}_K \dot{\subseteq} N_B$.

On the other hand, N_B being perfect, for any r we have $U_{U_r} = U_r$ which (again by the strong Markov property) implies $[U_r] \dot{\subseteq} M_K$ and consequently $N_B \dot{\subseteq} \overline{M}_K$.

Thus we have $N_B \doteq \overline{M}_K$ and the fact that $K = K^r$ is now a consequence of the equality $U = T^K$ a.s.

Finally, the set $M_{B^r \setminus K}$ may be written as the union of $M_{B^r} \setminus \overline{M}_K$ and $(\overline{M}_K \setminus M_K) \cap M_{B^r}$, both $\mathscr{K}$-optional and with countable sections.

Note that for any $\Gamma \in \mathscr{E}^{(n)}$ satisfying $\Gamma = \Gamma^r$ the set $\overline{M}_\Gamma$ is a.s. perfect. This together with the definition of K implies that any $C \subseteq B^r$ satisfying $C^r = C$ has the property that $C \subseteq K$.

§ 4. Local Times for Sets Satisfying $B = B^r$

With each additive functional A we associate the set $C(A)$ of its points of increase and the process

$$S_t^A(\omega) = \inf\{u > t:\ (u,\omega) \in C(A)\}.$$

Note that

$$S^A(\omega) = S_\alpha^A(\omega) = \inf\{u > \alpha(\omega):\ (u,\omega) \in C(A)\}$$
$$= \inf\{u > 0:\ A_u(\omega) > 0\}.$$

Definition 4.1. The additive functional $(A_t)_t$ is said to be a *local time* for the set $B \in \mathscr{E}^{(n)}$ if $\overline{M}_B \doteq C(A)$.

Since we are always looking for continuous local times and since the set of points of increase of a continuous additive functionals is perfect, this (in view of § 3) justifies the restriction to sets satisfying $B = B^r$.

Theorem 4.1. *If $B\in\mathscr{E}^{(n)}$ is such that $B=B^r$, then for each $\lambda\in(0,\infty)$ there exists a unique $\mathscr{K}$-predictable local time $(L_t^{(\lambda)})_t$ for B satisfying*

$$u^{\lambda}_{L^{(\lambda)}}(t,x)=f^{\lambda}_B(t,x) \quad \text{for all } (t,x)\in E.$$

The local time $(L_t^{(\lambda)})_t$ is a.s. continuous if and only if the process $\{T_t^B\}_t$ is quasi-left-continuous.

Proof. The λ-excessive function $f^{\lambda}_B(t,x)$ is a λ-potential of class (D) and therefore, according to Theorem 2.2, there exists a unique $\mathscr{K}$-predictable additive functional $(L_t^{(\lambda)})_t$ such that $u^{\lambda}_{L^{(\lambda)}}(t,x)=f^{\lambda}_B(t,x)$, for all $(t,x)\in E$. We are going to show that $(L_t^{(\lambda)})_t$ is a local time for B.

Since in the proof the set B, the constant λ and the corresponding additive functional $L^{(\lambda)}$ are fixed, we shall simply write T_t, T, L, C, S_t, S instead of T_t^B, T^B, $L^{(\lambda)}$, $C(L^{(\lambda)})$, $S_t^{L^{(\lambda)}}$, $S^{L^{(\lambda)}}$.

Let us first show that $S\geqq T$ a.s. on $\{T<\beta\}$. Applying the strong Markov property at T we get

$$P_{tx}\left\{\int_T^{\beta} e^{-\lambda(u-t)}dL_u;\, T<\beta\right\}=P_{tx}\{e^{-\lambda(T-t)}u^{\lambda}_L(T,x_T),\, T<\beta\}=f^{\lambda}_B(t,x)$$

for any $(t,x)\in E$; it follows that $\int_{\alpha}^{T} e^{-\lambda(u-\alpha)}dL_u=0$ a.s. on $\{T<\beta\}$ and hence $S\geqq T$ a.s. on $\{T<\beta\}$.

Next, since for any $(t,x)\in B$ we have

$$u^{\lambda}_L(t,x)=P_{tx}\{e^{-\lambda(S-t)}u^{\lambda}_L(S,x_S);\, S<\beta\}=1$$

we get that $P_{tx}(S=t)=1$ for any $(t,x)\in B$.

Suppose there exists $(s,y)\in E$ such that $P_{sy}(S>T;\, T<\infty)>0$; this implies $P_{sy}\{e^{-\lambda(S-T)};\, T<\infty\}<P_{sy}\{T<\infty\}$. On the other hand, since on $S>T$ we have $S_T=S$, by applying the strong Markov property at T we get

$$\begin{aligned}
&P_{sy}\{e^{-\lambda(S-T)};\, T<\infty\}\\
&\quad=P_{sy}\{S=T;\, T<\infty\}+P_{sy}\{e^{-\lambda(S_T-T)};\, S>T;\, T<\infty\}\\
&\quad=P_{sy}\{S=T;\, T<\beta\}\\
&\qquad+P_{sy}\{P_{T(\omega)xT(\omega)}\{\exp[-\lambda(S_{T(\omega)}(\omega')-T(\omega)]\};\, S(\omega)>T(\omega);\, T(\omega)<\infty\}\\
&\quad=P_{sy}\{S\geqq T;\, T<\beta\}=P_{sy}\{T<\beta\}.
\end{aligned}$$

Thus we must have $S=T$ a.s. on $\{T<\infty\}$. In fact we have $S=T$ a.s., since $u^{\lambda}_L(s,x)=P_{sx}\{e^{-\lambda(T-s)};\, T<\infty\}$ and $S=T$ a.s. on $\{T<\infty\}$ easily imply $P_{sy}\{S<\infty;\, T=\infty\}=0$, for each $(s,x)\in E$.

Taking now into account that for any t, $(T_t,x_{T_t})\in B$ a.s. on $T_t\in[\alpha,\beta)$ and that $T=S$ a.s., by applying the strong Markov property at T_t we get $S_{T_t}=T_t$ a.s. on $T_t\in[\alpha,\beta)$, which obviously implies $\overline{M}_B\dot{\subseteq}C$.

Now since $C\dot{\supseteq}\overline{M}_B$ we have $T_t\geqq S_t$ a.s. on $t\in[\alpha,\beta)$ and $P\{T_t>S_t;t\in[\alpha,\beta)\}=P\{P_{tx_t}(T_t>S_t);\, t\in[\alpha,\beta)\}=0$. Hence $T_t=S_t$ a.s. on $t\in[\alpha,\beta)$ and therefore $C\doteq\overline{M}_B$.

The fact that $(L_t)_t$ is a.s. continuous if and only if $(T_t)_t$ is quasi-left-continuous is an immediate consequence of the representation theorem.

Notes. 1. In fact in the proof of the theorem we have shown also that

$$\text{Supp}(L^{(\lambda)}) = \{(t,x): P_{tx}(S^{L^{(\lambda)}} = t) = 1\} = B.$$

2. We are now going to give a sufficient condition for the continuity of the local times obtained in Theorem 4.1., following and argument from [7].

For any $a>0$ consider the process

$$G_t^a = \inf\{u > t; T_u^B - u > a\}$$

which obviously belongs to Q. If

$$\lim_{\varepsilon \downarrow 0} \sup_{(t,x)\in B} P_{tx}(G_t^a - t \leqq \varepsilon) = 0$$

then $(L_t^{(\lambda)})_t$ is a.s. continuous.

Indeed, the points of discontinuity of $(T_t^B)_t$ are the left ends of the interval contiguous to $\overline{M}_B$ and therefore when checking the quasi-left-continuity of $(T_t^B)_t$ we have to consider only those stopping times U, that might appear among these extremities. Let U be such a stopping time, $(U_n)_n$ and increasing sequence of stopping times $(U_n)_n$ converging to U and $V_n = T^B_{U_n}$, for each n.

Applying the strong Markov property at V_n and taking into account that $(V_n, x_{V_n}) \in B$ a.s. on $V_n \in [\alpha, \beta)$ we get for each $a>0$

$$\begin{aligned} P\{T_U^B - U > a, V_n \uparrow U\} &= \lim_{\varepsilon \downarrow 0} \lim_{n\to\infty} P\{T_U^B - U > a, U - V_n \leqq \varepsilon\} \\ &\leqq \lim_{\varepsilon \downarrow 0} \lim_{n\to\infty} P\{G^a_{V_n} - V_n \leqq \varepsilon, \alpha \leqq V_n < \beta\} \\ &\leqq \lim_{\varepsilon \downarrow 0} \sup_{(t,x)\in B} P_{tx}\{G_t^a - t \leqq \varepsilon\} = 0 \end{aligned}$$

and this clearly shows that the imposed condition implies the quasi-left-continuity of the process $\{T_t^B\}_t$.

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Asymptotics for Interacting Particle Systems on Z^d

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Summary. Two of the simplest interacting particle systems are the coalescing random walks and the voter model. We are interested here in the asymptotic density and growth of these systems as $t \to \infty$. More specifically, let $(\xi_t^{Z^d})$ be a system of coalescing random walks with initial state Z^d, and (ζ_t^O) a voter model with a single individual originating at O. We analyse $p_t = P(O \in \xi_t^{Z^d}) = \hat{P}(\zeta_t^O \neq \emptyset)$, and show that $p_t \sim \frac{1}{\pi}\frac{\log t}{t}$ as $t \to \infty$ for $d=2$, and $p_t \sim (\gamma_d t)^{-1}$ as $t \to \infty$ for $d \geqq 3$ for some γ_d. As a consequence, conditioned on non-extinction of ζ_t^O, $p_t|\zeta_t^O|$ approaches an exponential distribution. Results of a recent paper by Sawyer are applied.

1. Introduction

Two of the simplest interacting particle systems on the d-dimensional integer lattice Z^d are the *coalescing random walks* $(\xi_t^{Z^d})$ and the *voter model* (ζ_t^O). The process $(\xi_t^{Z^d})$ consists of particles, one starting from each site $x \in Z^d$. These particles undergo independent continuous time rate one simple random walks, except that whenever a particle jumps to a site which is already occupied by another, then the two particles coalesce into one. The state space is $S = \{\text{all subsets of } Z^d\}$, where $x \in \xi_t$ if there is a particle at x at time t. The process (ζ_t^O) is a continuous time Markov chain on the denumerable space $S_0 = \{\text{finite subsets of } Z^d\}$, with $\zeta_0^O = \{O\}$; it executes the jumps

$$\begin{aligned} &A \to A \cup \{x\} \quad (x \notin A) \quad \text{at rate } \frac{1}{2d}|\{y \in A\colon \|y-x\|=1\}| \\ &A \to A - \{x\} \quad (x \in A) \quad \text{at rate } \frac{1}{2d}|\{y \in A^c\colon \|y-x\|=1\}|, \end{aligned} \tag{1}$$

$A \in S_0$, $x \in Z^d$. (We put $|B| =$ cardinality of $B \in S_0$, $A^c = Z^d - A$, $\| \ \| =$ Euclidean norm.) Here ζ_t^O may be thought of as the sites occupied by particles at time t, so

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that (ζ_t^O) represents the evolution of a finite configuration of particles on Z^d. The process $(\xi_t^{Z^d})$ belongs to the family of coalescing random walks $\{(\xi_t^A);\ A\in S\}$ (the superscript A denoting the initial state), a Markov family of S-valued processes. Similarly, (ζ_t^O) is one of the voter models in $\{(\zeta_t^A);\ A\in S\}$, another such family. These are among the most widely studied interacting particle systems; for references and a recent survey, see [9]. Both $\{(\xi_t^A)\}$ and $\{(\zeta_t^A)\}$ are *additive* in the sense of Harris [10]. Consequently the entire family $\{(\xi_t^A)\}$ can be constructed on a single canonical probability space and the family $\{(\zeta_t^A)\}$ on another probability space, with the aid of their respective *percolation substructures*, in such a way that additivity holds. Namely

$$\xi_t^{A\cup B}=\xi_t^A\cup\xi_t^B \qquad A,B\in S,\ t\geqq 0, \tag{2}$$

$$\zeta_t^{A\cup B}=\zeta_t^A\cup\zeta_t^B \qquad A,B\in S,\ t\geqq 0. \tag{3}$$

Moreover, the substructures for these two systems are *dual:* for P governing $\{(\xi_t^A)\}$ and $\hat{P}$ governing $\{(\zeta_t^A)\}$, a *duality equation* asserts that

$$P(\xi_t^A\cap B\neq\emptyset)=\hat{P}(\zeta_t^B\cap A\neq\emptyset) \qquad A,B\in S,\ t\geqq 0. \tag{4}$$

(See [9, 10] or [11] for more details concerning (2)–(4).) Setting $A=Z^d$ and $B=\{O\}$ in (4), we obtain

$$P(O\in\xi_t^{Z^d})=\hat{P}(\zeta_t^O\neq\emptyset); \tag{5}$$

thus the "particle density" of $\xi_t^{Z^d}$ equals the "survival probability" of ζ_t^O. Since we will be focussing on these two processes, from now on we will abbreviate $\xi_t=\xi_t^{Z^d}$, $\zeta_t=\zeta_t^O$. It follows easily from (1) that

$$n_t=|\zeta_t| \tag{6}$$

is a martingale. In fact, if τ_i is the time of the i'th jump of the process, then (n_{τ_i}) is a simple symmetric random walk with absorption at 0. The holding times $\tau_i-\tau_{i-1}$, however, are determined by the "boundary size" of ζ_{τ_i}, i.e., the length of the borderline curve between sites in ζ_{τ_i} and its complement. Since the geometry of ζ_{τ_i} is not at all obvious, detailed analysis of n_t is considerably more complicated. Nevertheless, since the jump is always at least 2, we do know that $|\zeta_t|$ is eventually absorved at 0 with probability one. Thus, if p_t denotes the common value in (5), i.e.,

$$p_t=P(O\in\xi_t)=\hat{P}(\zeta_t\neq\emptyset), \tag{7}$$

it follows that

$$p_t\downarrow 0 \quad \text{as } t\to\infty. \tag{8}$$

Our principal objective in this paper is to determine the exact asymptotics for p_t in every dimension d.

By way of motivation, let us first consider the one-dimensional case, where the analysis is quite easy. If $d=1$, then $(|\zeta_t^O|)$ *is* a rate-2 simple random walk on Z with absorption at 0, since (ζ_t^O) remains a "block" with boundary size 2 until it is

trapped at $\emptyset$. The reflection principle and local central limit theorem together imply that

$$p_t \sim \frac{1}{\sqrt{\pi t}} \quad \text{as } t \to \infty,$$

and

$$\lim_{t\to\infty} P(n_t \leqq \alpha E[n_t | n_t > 0] \,|\, n_t > 0) = 1 - e^{-\frac{\pi}{4}\alpha^2}, \tag{9}$$

$\alpha \in [0, \infty)$. Using the joint percolation substructure for $\{(\xi_s^A)\}$ and $\{(\zeta_s^A)\}$, $0 \leqq s \leqq t$ (cf. [9] or [10]), one sees that n_t also has an interpretation for the coalescing random walks. Namely (in any dimension d),

$$\begin{aligned} n_t = &\text{the number of walks in } (\xi_\cdot) \text{ which are at the origin} \\ &\text{at time } t \text{ as a result of coalescence.} \end{aligned} \tag{10}$$

Thus (9) is a weak convergence result for the number of collisions up to time t experienced by the particle at 0, given that 0 is occupied at time t. Alternatively, if we think of the masses of particles as added together upon coalescing, then (9) describes the distribution of masses.

In dimension $d \geqq 2$, matters are not so simple. Here the evolution of the boundary of (ζ_t) seems unmanageable; one must therefore resort to indirect reasoning to investigate p_t. In [4], the fact that the boundary of ζ_t must have dimension at least $d-1$ was used to establish the crude estimate:

$$p_t = O(t^{-d/(d+1)}) \quad \text{as } t \to \infty.$$

To date, this is apparently the only available upper bound. A different approach to the study of (ζ_t) has relied on comparison with a certain "multi-type voter process" $(\bar{\zeta}_t)$, about which more has been proved. $(\bar{\zeta}_t) = ((\bar{\zeta}_t(x);\ x \in Z^d))$ has state space $(Z^d)^{Z^d}$, and may be defined in terms of the percolation substructure $\hat{\mathscr{P}}$ for $\{(\zeta_t^A)\}$ by

$$\bar{\zeta}_t(x) = y \quad \text{if there is a path up from } (y, 0) \text{ to } (x, t) \text{ in } \hat{\mathscr{P}} \tag{11}$$

(cf. [9] or [10]). $(\bar{\zeta}_t)$ is well-defined because there is always exactly one y which works in (11). Introduce

$$N_t = |\{x \in Z^d : \bar{\zeta}_t(x) = \bar{\zeta}_t(O)\}|.$$

That is, let N_t be the "patch size" of particles in the multitype process which are the same type as the one at the origin at time t. Sudbury [17] observed that if R_t is the range of a (continuous time rate 1) simple random walk on Z^d up to time t, then

$$\hat{E}[N_t] = E[R_{2t}]. \tag{12}$$

Subsequently, Kelly [12] noted that

$$\hat{P}(N_t = j) = j\hat{P}(n_t = j) \quad j = 0, 1, \ldots, \tag{13}$$

so that

$$p_t = \hat{P}(n_t > 0) = \hat{E}[N_t^{-1}].$$

Asymptotics for $E[R_t]$ were obtained in a celebrated paper by Dvoretsky and Erdös [8]; their results, together with (12), yield

$$\begin{aligned} \hat{E}[N_t] &\sim 2\pi \frac{t}{\log t} \quad \text{as } t \to \infty \quad d=2, \\ &\sim 2\gamma_d t \qquad \text{as } t \to \infty \quad d \geqq 3, \end{aligned} \tag{14}$$

where γ_d is the probability that d-dimensional simple random walk never returns to its initial position. Kelly [12] exhibited lower bounds for p_t by noting that

$$p_t = \hat{E}[N_t^{-1}] \geqq (\hat{E}[N_t])^{-1} \tag{15}$$

and then appealing to (14).

More recently, Sawyer has extended this line of analysis in a remarkable paper [14]. His results deal with a generalization of the multi-type voter model which is known to mathematical geneticists as the "stepping stone model". He is able to compute the asymptotics of all moments for the size of the patch containing the origin in this model. It follows that the patch size, suitably normalized, converges weakly to a Γ-distributed limit. Specialized to (ζ_t), the result is as follows.

Sawyer's Theorem. *For $d \geqq 2$,*

$$\lim_{t \to \infty} \hat{E}\left[\left(\frac{N_t}{E[N_t]}\right)^k\right] = \frac{(k+1)!}{2^k} \qquad k = 1, 2, \ldots . \tag{16}$$

Thus for any $\alpha \in [0, \infty)$,

$$\lim_{t \to \infty} \hat{P}(N_t \leqq \alpha \hat{E}[N_t]) = \int_0^\alpha 4ue^{-2u}du,$$

where $\hat{E}[N_t]$ satisfies (14).

Sawyer's Theorem comes tantalizingly close to determining the asymptotics for p_t. From (13) we see that

$$\hat{E}[n_t^k] = \hat{E}[N_t^{k-1}] \qquad k \geqq 1, \tag{17}$$

so we now asymptotics for all the moments of n_t. Unfortunately, this is not quite enough to handle $p_t = P(n_t > 0) = E[N_t^{-1}]$. There is a problem of "tightness near 0"; comparatively small values of N_t may have a drastic influence on p_t without affecting (17) in the limit.

The essential objective in this paper is to bridge the "technical gap" in the approach just outlined. In so doing, we derive exact asymptotics for p_t, as well as a counterpart to Sawyer's weak convergence theorem. Our main result is

Theorem 1'. *Let (ξ_t) be the d-dimensional coalescing random walks starting from Z^d, (ζ_t) the voter model on Z^d starting with a single particle at O. If p_t is given by* (7), *and n_t by* (6) *or* (10), *then*

$$\begin{aligned} p_t &\sim \frac{1}{\pi} \frac{\log t}{t} \quad \text{as } t \to \infty \quad d=2 \\ &\sim (\gamma_d t)^{-1} \quad \text{as } t \to \infty \quad d \geqq 3 \end{aligned} \tag{18}$$

(γ_d *is as in* (14)). *For any* $d \geqq 2$, $\alpha \in [0, \infty)$,

$$\lim_{t \to \infty} P(n_t \leqq \alpha p_t^{-1} | n_t > 0) = 1 - e^{-\alpha}. \tag{19}$$

We are unable to prove Theorem 1 directly from moment considerations. Rather, our approach is to obtain *good* upper bounds for p_t:

Theorem 1.

$$\begin{aligned} p_t &= O\left(\frac{\log t}{t}\right) \quad \text{as } t \to \infty \quad d = 2, \\ &= O(t^{-1}) \quad \text{as } t \to \infty \quad d \geqq 3. \end{aligned} \tag{20}$$

We then apply Sawyer's Theorem in order to determine the correct asymptotic constant. Once this constant is known, the accompanying weak convergence result follows easily.

Section 2 contains the proof that Theorem 1 in conjunction with Sawyer's Theorem yields Theorem 1′. In Sect. 3 we prove Theorem 1. It is instructive to note that the proof deals almost entirely with the infinite system of coalescing random walks. The conventional wisdom of duality theory is that the finite dual system $\{(\zeta_t^A);\ A \in S_0\}$ is simpler to analyse than the infinite system $\{(\xi_t^A);\ A \in S\}$. But the finite voter model in more than one dimension turns out to be sufficiently complicated that the ultimate solution of our problem involves considerable interplay between $\{(\xi_t^A)\}$, $\{(\zeta_t^A)\}$ and $(\bar{\zeta}_t)$. Finally, Sect. 4 contains some additional remarks and open problems.

2. Theorem 1 Implies Theorem 1′

In this section we show, with the aid of Sawyer's Theorem, that (20) implies (18) and (19). The proof is not difficult; for the sake of clarity we make use of two lemmas. Throughout the remainder of the paper the dimension d will be thought of as fixed, and usually suppressed in the notation. It will also be convenient to introduce the notation

$$\begin{aligned} f_t &= \frac{t}{\log t} \quad d = 2, \\ &= t \quad d \geqq 3, \\ g_t &= f_t p_t, \end{aligned}$$

and

$$\begin{aligned} K &= \pi \quad d = 2, \\ &= \gamma_d \quad d \geqq 3 \ (\gamma_d \text{ is as in (14)}). \end{aligned}$$

We now demonstrate the two lemmas.

Lemma 1. *For any* $\varepsilon > 0$,

$$\lim_{t \to \infty} K f_t \hat{P}(n_t > \varepsilon K f_t) = e^{-\varepsilon}.$$

Proof. By (13), the left side equals

$$\tfrac{1}{2} \hat{E}\left[\left(\frac{N_t}{2Kf_t}\right)^{-1}, \frac{N_t}{2Kf_t} > \frac{\varepsilon}{2}\right]. \tag{21}$$

For N a random variable with density $4ue^{-2u}$, Sawyer's weak convergence result, together with bounded convergence, implies that as $t \to \infty$, (21) converges to

$$\tfrac{1}{2}E\left[N^{-1}, N > \frac{\varepsilon}{2}\right] = \tfrac{1}{2}\int_{\frac{\varepsilon}{2}}^{\infty} \frac{1}{u} 4ue^{-2u}\,du$$
$$= e^{-\varepsilon}. \quad \square$$

Lemma 2. *If $g_t = O(1)$ as $t \to \infty$, then*

$$\lim_{\varepsilon \to 0} \limsup_{t \to \infty} f_t \hat{P}(n_t \in (0, \varepsilon f_t]) = 0.$$

Proof. For any $s \leqq t$,

$$\hat{P}(n_t \in (0, \varepsilon f_t]) = \hat{P}(n_s \in (0, \varepsilon f_t], n_t > 0) + \hat{P}(n_s > \varepsilon f_t, n_t > 0) - \hat{P}(n_t > \varepsilon f_t).$$

Taking $s = (1 - \sqrt{\varepsilon})t$, and dropping $n_t > 0$ from the middle term on the right, we have

$$f_t \hat{P}(n_t \in (0, \varepsilon f_t]) \leqq f_t \hat{P}(n_{(1-\sqrt{\varepsilon})t} \in (0, \varepsilon f_t],\ \zeta_t \neq \emptyset) + f_t \left| \hat{P}(n_{(1-\sqrt{\varepsilon})t} > \varepsilon f_t) - \hat{P}(n_t > \varepsilon f_t) \right|.$$

By Lemma 1, the second term on the right is

$$O\left(\left| \frac{1}{1-\sqrt{\varepsilon}} e^{-\frac{\varepsilon}{K(1-\sqrt{\varepsilon})}} - e^{-\frac{\varepsilon}{K}} \right| \right) \quad \text{as } t \to \infty,$$

and hence tends to 0 as $\varepsilon \to 0$, uniformly for large t. To bound the first term we use the Markov property, (3), and translation invariance:

$$\begin{aligned} f_t \hat{P}(n_{(1-\sqrt{\varepsilon})t} \in (0, \varepsilon f_t],\ \zeta_t \neq \emptyset) &= f_t \sum_{A:\, |A| \in (0, \varepsilon f_t]} \hat{P}(\zeta_{(1-\sqrt{\varepsilon})t} = A)\, \hat{P}(\zeta^A_{\sqrt{\varepsilon}t} \neq \emptyset) \\ &= f_t \sum_{A:\, |A| \in (0, \varepsilon f_t]} \hat{P}(\zeta_{(1-\sqrt{\varepsilon})t} = A)\, \hat{P}\Big(\bigcup_{x \in A} \zeta^x_{\sqrt{\varepsilon}t} \neq \emptyset\Big) \\ &\leqq f_t p_{(1-\sqrt{\varepsilon})t} \varepsilon f_t p_{\sqrt{\varepsilon}t} \\ &\leqq \frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}} g_{(1-\sqrt{\varepsilon})t}\, g_{\sqrt{\varepsilon}t}. \end{aligned}$$

By hypothesis this last term is $\dfrac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}[O(1)]^2$, and therefore also tends to 0 uniformly for large t as $\varepsilon \to 0$. $\square$

Lemma 2 establishes the "tightness at 0" necessary to apply Sawyer's Theorem to our problem.

Proposition. *If $g_t = O(1)$ as $t \to \infty$, then* (18) *and* (19) *hold. Hence* (18) *and* (19) *follow from* (20).

Proof. By Lemma 1,

$$\liminf_{t\to\infty} Kg_t \geqq \lim_{\varepsilon\to 0}\lim_{t\to\infty} Kf_t\hat{P}(n_t>\varepsilon Kf_t)=1.$$

Also, for any $\varepsilon>0$,

$$\begin{aligned}\limsup_{t\to\infty} Kg_t &\leqq \lim_{t\to\infty} Kf_t\hat{P}(n_t>\varepsilon Kf_t)+\limsup_{t\to\infty} Kf_t\hat{P}(n_t\in(0,\varepsilon Kf_t])\\ &=e^{-\varepsilon}+\delta_\varepsilon,\end{aligned}$$

where $\delta_\varepsilon\to 0$ as $\varepsilon\to 0$ by Lemma 2. Thus $\limsup_{t\to\infty} Kg_t=1$. We conclude that

$$p_t\sim(Kf_t)^{-1} \quad \text{as } t\to\infty,$$

i.e., (18) holds. Now, using (16)–(18), observe that

$$\begin{aligned}\hat{E}\left[\left(\frac{n_t}{Kf_t}\right)^k\middle| n_t>0\right] &= \hat{E}\left[\left(\frac{N_t}{2Kf_t}\right)^{k-1}\right]2^{k-1}\\ &\to 2^{k-1}\hat{E}[N^{k-1}]=2^{k-1}\frac{k!}{2^{k-1}}=k! \quad \text{as } t\to\infty.\end{aligned}$$

Since $\hat{E}[n_t|n_t>0]=p_t^{-1}\sim Kf_t$, and since $k!$ is the k'th moment of the standard exponential distribution, (19) follows by the method of moments. □

3. Proof of Theorem 1

Our objective in this section is to prove

$$\begin{aligned}p_t &= O\left(\frac{\log t}{t}\right) \quad \text{as } t\to\infty \quad d=2,\\ &= O(t^{-1}) \quad \text{as } t\to\infty \quad d\geqq 3.\end{aligned} \tag{20}$$

The proof constitues the major effort of the paper. We will work directly with the infinite systems of coalescing random walks $\{(\xi_t^A);\ A\in S\}$; since the proof is somewhat long, we pause to introduce some needed terminology.

Extensive use will be made of the graphical representation of $\{(\xi_t^A)\}$, by means of which the entire family is defined on a single probability space; thus a certain familiarity with [9] or [10] will be assumed. $\{(\xi_t^A)\}$ is induced by its percolation substructure $\mathscr{P}$:

$$\begin{aligned}&\xi_t^A=\{x\in Z^d\colon N_t^A(x)>0\} \quad A\in S,\ t\geqq 0,\\ &\text{where } N_t^A(x) \text{ is the number of paths up from } (A,0) \text{ to } (x,t) \text{ in } \mathscr{P}.\end{aligned} \tag{22}$$

More generally, if $N_{t,u}^A(B)$ denotes the number of paths up from (A,t) to (B,u) $(t\leqq u)$, then we can define

$$\xi_{t,u}^A=\{x\in Z^d\colon N_{t,u}^A(x)>0\} \quad A\in S,\ 0\leqq t\leqq u<\infty.$$

Note that

$$\xi_{t,u}^{\xi_t^A} = \xi_u^A \quad 0 \leqq t \leqq u < \infty. \tag{23}$$

Also, the additivity property (2) extends to

$$\xi_{t,u}^{A \cup B} = \xi_{t,u}^A \cup \xi_{t,u}^B \quad A, B \in S,\ 0 \leqq t \leqq u < \infty. \tag{24}$$

For brevity's sake, we write $\xi_{t,u} = \xi_{t,u}^{Z^d}$, $\xi_t = \xi_t^{Z^d}$.

We let Λ denote the *box of side* $2R$ *centered at* O, i.e.,

$$\Lambda = \{(x_1, \ldots, x_d) \in Z^d : |x_i| \leqq R\},$$

where R is a positive integer. Also, let V be the unique collection of sites in Z^d such that the translations $\Lambda + v$, $v \in V$, partition Z^d. Any choice of R gives rise to a corresponding Λ and V. In what follows, think of R, Λ and V as fixed.

We are now prepared to begin estimation of p_t. As the motivation may tend to become buried in the construction we are to employ, we first present a brief outline. We wish to show that p_t is decreasing at at least a fixed rate: $\log t/t$ if $d = 2$ and $1/t$ if $d \geqq 3$. Our basic approach will be to exploit the fact that the larger p_t is, the more rapidly it must decay: the more coalescing random walks present at time t, the more frequently they coalesce with one another, and the more rapidly their density decreases. Thus, the process has, in some sense, a self-correcting mechanism which limits the number of distinct particles. The upper estimates obtained in this manner for p_t are (perhaps surprisingly) strong enough to yield (20).

The methodology may be considered in analogy with a vector field (or flow) diagram in the context of differential equations. The function p_t has a negative derivative at each point (t, p_t) with $p_t > 0$, and becomes increasingly negative as p_t increases. If it is also true that $g_t = f_t p_t$ (as defined in Sect. 2) has a negative derivative at large enough values of g_t, then g_t will be bounded, which is sufficient for (20). The actual problem is, however, somewhat more complicated in that it does not seem possible to compute p_t' itself. For a given density of particles at some fixed time in the coalescing random walk system, it is not clear what the actual spatial distribution of particles is; the rate at which particles coalesce is however certainly dependent on this distribution. (For example, if the particles are all spread far apart, then this rate will be zero.) Yet, one may circumvent this difficulty by choosing to observe the coalescing random walk system over a fixed spatial region for a time interval of appropriate length.

What we do is to compute the decrease of the particle density over a box Λ with side $2R$. If the box is chosen large enough, it will at time t contain a minimal expected number of particles (two is sufficient for our purposes). In Lemma 3, it is shown that, no matter what the relative positions of the particles in the box, then after additional time s, a minimal number of these particles will have coalesced. Repeating this procedure over disjoint time intervals, we obtain the estimate (27), which bounds g_t in terms of this minimal expectation. This minimal rate at which particles coalesce may be computed without difficulty (Lemma 5). Therefore, appropriate choice of the spatial and time scales R and s (Lemma 4) ensures the desired bound given by (20).

We start off by reformulating the problem slightly, and define

$$e_t(B)=E[|\xi_t\cap B|] \qquad B\in S_0,\ t\geqq 0.$$

Note that by translation invariance

$$e_t(B)=|B|\,p_t.$$

Thus, a little algebra shows that whenever $0\leqq t\leqq u<\infty$,

$$p_u=p_t\left[1-\frac{e_t(\Lambda)-e_u(\Lambda)}{e_t(\Lambda)}\right]. \tag{25}$$

On the other hand, (23), (24) and translation invariance yield

$$\begin{aligned} e_u(\Lambda)&=\sum_{x\in\Lambda} P(x\in\xi_{t,u}^{\xi_t})\\ &\leqq \sum_{x\in\Lambda}\sum_{v\in V} P(x\in\xi_{t,u}^{\xi_t\cap\Lambda+v})\\ &=\sum_{x\in\Lambda}\sum_{v\in V} P(x-v\in\xi_{t,u}^{\xi_t\cap\Lambda})\\ &=E[|\xi_{t,u}^{\xi_t\cap\Lambda}|]. \end{aligned}$$

Substituting this into (25), we obtain

$$p_u\leqq p_t\left[1-\frac{\Delta_{t,u}(\Lambda)}{e_t(\Lambda)}\right], \qquad \text{where } \Delta_{t,u}(\Lambda)=E[|\xi_t\cap\Lambda|-|\xi_{t,u}^{\xi_t\cap\Lambda}|]. \tag{26}$$

Inequality (26) states that p_t decreases at at least some fixed rate which involves the quantity $\Delta_{t,u}(\Lambda)$. The following lemma enables us to make this expression more explicit, and states that particles executing coalescing random walks coalesce at at least a minimal rate independent of their initial configuration.

Lemma 3. *Let P_x be the probability law governing continuous time rate 2 d-dimensional simple random walk starting from x. If τ is the hitting time for the origin, denote $H_s(x)=P_x(\tau<s)$. Let $\{(\xi_t^A)\}$ be the (rate 1) coalescing random walks on Z^d. Then for any $B\in S_0$, $B\neq\emptyset$,*

$$|B|-E[|\xi_s^B|]\geqq(|B|-1)\min_{x,\,y\in B} H_s(y-x).$$

Proof. The left side is the expected number of times particles of (ξ_s^B) coalesce through time s. The right side should be thought of as a lower bound for the expected number of particles which coalesce with some fixed particle through time s, where all interaction among other particles is suppressed. Formally, we note that in the percolation substructure $\mathscr{P}$ for $\{(\xi_t^A)\}$,

$$N_t^B(Z^d)=|B| \qquad B\in S_0,\ t\geqq 0.$$

($N_t^B(Z^d)$ is as in (22).) So for any $x\in B$,

$$\begin{aligned} |B|-|\xi_s^B|&=N_s^B(Z^d)-|\{z\in Z^d\colon N_s^B(z)>0\}|\\ &=N_s^B(\xi_s^x)+N_s^B(Z^d-\xi_s^x)-(1+|\{z\in Z^d-\xi_s^x\colon N_s^B(z)>0\}|)\\ &\geqq N_s^B(\xi_s^x)-1. \end{aligned}$$

Taking expectations, we see that

$$|B| - E[|\xi_s^B|] \geqq \sum_{\substack{y \in B \\ y \neq \hat{x}}} P(\xi_s^y = \xi_s^{\hat{x}}).$$

Since the distance between ξ_t^y and $\xi_t^{\hat{x}}$ is a rate 2 simple random walk, the claim follows. □

As a consequence of the Markov property and Lemma 3, we see that

$$\begin{aligned} \Delta_{t,u}(\Lambda) &= \sum_{B \subset \Lambda} P(\xi_t \cap \Lambda = B)\, E[|B| - |\xi_{t,u}^B|] \\ &\geqq \sum_{\substack{B \subset \Lambda \\ B \neq \emptyset}} P(\xi_t \cap \Lambda = B)(|B| - 1) \min_{x,y \in B} H_{u-t}(y-x) \\ &\geqq (e_t(\Lambda) - 1) \min_{x,y \in \Lambda} H_{u-t}(y-x). \end{aligned}$$

We also let $u = t+s$, and set

$$h_s = \min_{x,y \in \Lambda} H_s(y-x).$$

Substitution into (26) then yields

$$p_{t+s} \leqq p_t[1 - (1 - [e_t(\Lambda)]^{-1}) h_s] \qquad s, t \geqq 0.$$

Thus, we now have a bound for the rate of decrease of p_t after additional time s. To make the bound completely explicit, we still have to choose Λ and s as functions of t. Given t, we choose $\Lambda = \Lambda_t$ to have side $2R$ with

$$R = R_t = \lceil (2p_t^{-1})^{\frac{1}{d}} \rceil$$

($\lceil k \rceil$ = least integer $\geqq k$), in order that

$$e_t(\Lambda) = R^d p_t \geqq 2.$$

With Λ so chosen we obtain the bound

$$p_{t+s} \leqq p_t[1 - \tfrac{1}{2} h_s] \qquad s, t \geqq 0.$$

To make this inequality somewhat easier to handle, iterate it a total of $\lfloor ts^{-1} \rfloor$ times ($\lfloor k \rfloor$ = greatest integer $\leqq k$). Using the fact that p_t is decreasing we conclude that

$$\begin{aligned} p_{2t} &= p_t[1 - \tfrac{1}{2} h_s]^{\lfloor t/s \rfloor} \\ &\leqq p_t \exp\{-\tfrac{1}{2} h_s \lfloor t/s \rfloor\} \qquad s, t \geqq 0. \end{aligned}$$

Since $f_{2t}/f_t \leqq 2$, this last inequality may be rewritten as

$$g_{2t} \leqq g_t \exp\{\log 2 - \tfrac{1}{2} h_s \lfloor t/s \rfloor\} \qquad s, t \geqq 0. \tag{27}$$

Our goal in this section is to demonstrate (20), namely, to show that g_t is uniformly bounded on $[0, \infty)$. Inequality (27) is our main tool. As a consequence

of the next two lemmas, it will follow that the argument of the exponential term above is negative for large values of g_t. Therefore by (27), g_t cannot become too large, but must remain bounded. We now proceed to make this reasoning precise.

Lemma 4. *For an appropriate choice of* $s=s_t$,

$$\liminf_{t\to\infty} g_t^{-1} h_{s_t}\lfloor t/s_t\rfloor > 0. \tag{28}$$

Assuming (28) for a moment, let us show that (20) follows from (27) and (28). By (28) there is an $\varepsilon>0$ and a $t_0<\infty$ such that

$$h_{s_t}\lfloor t/s_t\rfloor > \varepsilon g_t \quad \text{for all } t\geqq t_0. \tag{29}$$

We claim that

$$g_t \leqq \Big(\max_{r\in[0,\,2t_0]} g_r\Big) \vee \frac{4\log 2}{\varepsilon} \quad \text{for all } t\geqq 0 \tag{30}$$

($a\vee b=\max\{a,b\}$). Suppose that (30) is violated at $t=t_1$, and let

$$t_* = \min\{2^{-n}t_1 : (30) \text{ is violated at } 2^{-n}t_1;\ n=0,1,\ldots\}. \tag{31}$$

g_{t_*} violates (30), so $t_*>2t_0$, and hence $t_*/2>t_0$. With ε as in (29), if $g_{t_*/2}>\dfrac{2\log 2}{\varepsilon}$, then (29) and (27) imply that

$$g_{t_*} < g_{t_*/2},$$

which contradicts (31). If, on the other hand, $g_{t_*/2}\leqq\dfrac{2\log 2}{\varepsilon}$, then since $f_{2t}/f_t\leqq 2$ and p_t decreases,

$$g_{t_1}\leqq g_{t_*}\leqq\frac{4\log 2}{\varepsilon}.$$

Thus (30) cannot be violated for any t, i.e., (20) holds.

To prove Lemma 4, thereby completing the proof of Theorem 1, we will need some elementary estimates for rate 2 continuous time simple random walk on Z^d. Let (X_t) denote such a process, P_x its law starting from x, $P_t(x,y)=P_x(X_t=y)$, $G_t(x)=\int_0^t P_s(x,0)\,ds$, and define $H_t(x)$ as in Lemma 3. Then we have the following lemma.

Lemma 5. *If* $x\in Z^d$ *with* $\|x\|=r$, *then there is a constant* $C_d>0$ *such that*

$$\begin{aligned} H_{r^2}(x) &\geqq C_2/\log r \qquad d=2\\ &\geqq C_d r^{2-d} \qquad d\geqq 3. \end{aligned}$$

Proof. Use the inequality

$$H_t(x)\geqq G_t(x)/G_t(0),$$

together with the familiar asymptotics for the Green's function: as $r=\|x\|\to\infty$,

$$\begin{aligned} G_{r^2}(x) &\sim \alpha_2 && d=2,\\ &\sim \alpha_d r^{2-d} && d\geqq 3;\\ G_{r^2}(0) &\sim \beta_2 \log r && d=2,\\ &\sim \sup_t G_t(0)=\beta_d && d\geqq 3; \end{aligned}$$

α_d and β_d are positive (finite) constants. The local central limit theorem yields these asymptotics (cf. the (discrete-time) computations in the proof of Proposition 26.1 of [16].) □

We now proceed to verify (28) for a suitable choice of s_t.

Proof of Lemma 4. Choose

$$s_t = d\lceil (2p_t^{-1})^{\frac{1}{d}} \rceil^2.$$

Since the distance between any two sites in Λ_t is at most $D_t = \sqrt{d} R_t = \sqrt{d}\lceil (2p_t^{-1})^{\frac{1}{d}} \rceil$, Lemma 5 implies that

$$\begin{aligned} h_{s_t} = \min_{x,y \in \Lambda_t} H_{s_t}(y-x) &\geqq \min_{x,y \in \Lambda_t} H_{\|x-y\|^2}(y-x) \\ &\geqq C_2/\log D_t \qquad d=2 \\ &\geqq C_d D_t^{2-d} \qquad d \geqq 3. \end{aligned}$$

Now (8), (14) and (15) imply that as $t \to \infty$

$$p_t^{-1} \to \infty \quad \text{and} \quad t p^{\frac{2}{d}} \to \infty,$$

so

$$\lceil (2p_t^{-1})^{\frac{1}{d}} \rceil \sim (2p_t^{-1})^{\frac{1}{d}}$$

and

$$\lfloor t/s_t \rfloor \sim t/s_t.$$

It follows that for $d=2$,

$$\begin{aligned} \liminf_{t \to \infty} g_t^{-1} h_{s_t} \lfloor t/s_t \rfloor &\geqq \liminf_{t \to \infty} \frac{\log t}{t} p_t^{-1} \cdot \frac{C_2}{\log (2p_t^{-\frac{1}{2}})} \cdot \frac{t}{4p_t^{-1}} \\ &\geqq \liminf_{t \to \infty} \frac{C_2}{4} \left[\frac{\log t}{\log 2 + \frac{1}{2} \log p_t^{-1}} \right] \\ &\geqq C_2/2 > 0, \end{aligned}$$

where we have used the fact that $\log t / \log p_t^{-1} \geqq 1$ for large t since $t p_t \to \infty$. Similarly, for $d \geqq 3$,

$$\begin{aligned} \liminf_{t \to \infty} g_t^{-1} h_{s_t} \lfloor t/s_t \rfloor &\geqq \liminf_{t \to \infty} t^{-1} p_t^{-1} \cdot C_d [\sqrt{d} (2p_t^{-1})^{\frac{1}{d}}]^{2-d} \frac{t}{d(2p_t^{-1})^{2/d}} \\ &= \frac{C_d}{2d^{d/2}} > 0. \end{aligned}$$

This completes the proof of Lemma 4, and hence of Theorem 1. □

4. Additional Remarks

We discuss briefly three directions in which Theorem 1′ may be extended.

(i) *Domains of attraction.* If the initial state Z^d is replaced by other infinite states $A \in S$, or more generally measures μ on S, what happens to (18)? The

answer, for a large class of "nice" states, is that *nothing* new happens, i.e., Theorem 1′ is quite stable under changes in the initial distribution. This phenomenon was studied in the one-dimensional setting in [3]. A simple illustration is the following result.

Theorem 2. *Let $(\xi_t^{\mu_\theta})$ be the coalescing random walks on Z^d starting from Bernoulli product measure with density $\theta\in[0,1]$. Let $p_t^\theta = P(O\in\xi_t^{\mu_\theta})$. If $\theta\in(0,1)$, then p_t^θ has the same asymptotics* (18) *as p_t.*

Proof. A special case of the duality equation for product measures (see e.g. (1.11) in [9]) asserts that

$$P(O\notin\xi_t^{\mu_\theta}) = \hat{E}[(1-\theta)^{|\zeta_t|}].$$

Manipulating this we get

$$0\leqq 1-\frac{p_t^\theta}{p_t} = \sum_{k=1}^{\infty} P(|\zeta_t|=k\,|\,\zeta_t\neq\emptyset)(1-\theta)^k.$$

Bound the rightmost expression by

$$P(|\zeta_t|\in(0,\varepsilon f_t]\,|\,\zeta_t\neq\emptyset)+(1-\theta)^{\varepsilon f_t},$$

where f_t is defined as in Section 2. If we let $t\to\infty$, then the second term goes to 0; also as $\varepsilon\to 0$, the first term goes to 0 uniformly in t by Theorems 1 and 1′ and Lemma 2. Consequently,

$$\lim_{t\to\infty}\frac{p_t^\theta}{p_t}=1,$$

which proves the theorem. □

(ii) *Annihilating random walks.* A second family of interacting particle systems closely related to $\{(\xi_t^A)\}$ is the annihilating random walks $\{(\eta_t^A)\}$, where particles annihilate one another rather than coalesce upon collision. The systems (η_t^A) tend to be more difficult to analyse than (ξ_t^A), but fortunately the asymptotic density starting from Z^d is determined by combining Theorem 1′ with a recent result due to Arratia [2].

Theorem 3. *Let (η_t) be d-dimensional annihilating random walks starting from Z^d, and let $\tilde{p}_t = P(O\in\eta_t)$. Then $\tilde{p}_t$ satisfies* (18) *with $\frac{1}{\pi}$ replaced by $\frac{1}{2\pi}$, γ_d replaced by $2\gamma_d$.*

Proof. A special case of Arratia's theorem states that

$$\lim_{t\to\infty}\frac{\tilde{p}_t}{p_t}=\tfrac{1}{2}.$$

(iii) *Shape of the voter model.* Theorem 1′ raises some intriguing and challenging open problems concerning the voter model. The process $(\zeta_t\,|\,\zeta_t\neq\emptyset)$ may be thought of as the limiting critical case of a one-parameter family of supercritical models for tumour growth introduced by Williams and Bjerknes [18]. The supercriticial models, conditioned on nonextinction, have an asymptotic

shape, which grows linearly in "radius" [5, 6]. By analogy, one can ask shape questions about $(\zeta_t | \zeta_t \neq \emptyset)$. The limit law (19) suggests that the normalized shape itself might converge in distribution in some sense. Any results along these lines would be of interest. As one would expect, the situation is altogether different from the supercritical case. For example, whereas the supercritical processes on Z^d have boundary of asymptotic dimension $d-1$, Theorem 1′ shows that $(\zeta_t | \zeta_t \neq \emptyset)$ has boundary of asymptotic dimension d for $d \geqq 3$, "nearly d" for $d=2$.

(iv) *Coalescing Brownian motions.* Closely related systems of coalescing Brownian motions have been studied by Arratia [1] in dimension one, and by Smoluchowski [15], Chandrasekhar [7] and Lang and Nguyen [13] in dimension three.

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Determination of Probability Measures Through Group Actions

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Introduction

Let $(X, \mathscr{B}_X)$ be a measurable space. A subclass $\mathscr{D}$ of $\mathscr{B}_X$ is called a *determining class* for a family $\mathscr{P}$ of probability measures on $(X, \mathscr{B}_X)$ if for $\mu, \nu \in \mathscr{P}$, whenever $\mu(B) = \nu(B)$ for every $B \in \mathscr{D}$, then $\mu(B) = \nu(B)$ for every $B \in \mathscr{B}_X$. The first basic theorem of Measure Theory, viz., the Caratheodory-Extension Theorem tells us that any sub-algebra $\mathscr{D}$ of $\mathscr{B}_X$ which generates $\mathscr{B}_X$ is a determining class for the family of all probability measures on $(X, \mathscr{B}_X)$. The problem of finding determining classes, other than the ones assured by the Caratheodory-Extension Theorem, has been of interest. One of the earlier results in this direction is due to Cramer and Wold [4]. They considered the case $X = \mathbb{R}^n$ and showed that the class $\mathscr{D} = \{S_{\underline{t},s} | \underline{t} \in \mathbb{R}^n, s \in \mathbb{R}\}$ is a determining class for the family of all probability measures on $\mathbb{R}^n$, where for $\underline{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and $s \in \mathbb{R}$, $S_{\underline{t},s} = \left\{ \underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \middle| \sum_{i=1}^{n} t_i x_i < s \right\}$. In the case when X is a metric space $\mathscr{B}_X$ is the σ-algebra of Borel subsets of X, various determining classes (which utilize the topology of X) are known for the family $\mathscr{M}(X)$ of all probability measures on $\mathscr{B}_X$. For example one knows that the class $\mathscr{O}$ of all open subsets of X is a determining class for $\mathscr{M}(X)$. If X is a complete separable metric space, then the class $\mathscr{K}$ of all compact subsets of X is a determining class for $\mathscr{M}(X)$. Another class of sets in a metric space, which is of interest, is the class $\mathscr{S}$ of all closed balls in X. It turns out that in general $\mathscr{S}$ is not a determining class for $\mathscr{M}(X)$. Davis [5] has shown that one can construct a compact metric space and two distinct probability measures on it which agree on all closed balls. The problem of finding metric spaces X in which the class $\mathscr{S}$ is a determining class for $\mathscr{M}(X)$ has been investigated by many authors, for example: Besicovitch [2], Anderson [1], Christensen [3] and Hoffmann-Jorgensen [7].

Utilizing the group structure of $\mathbb{R}^n$, a method for constructing determining classes in $\mathbb{R}^n$ was given by Sapogov [10]. Let $E \in \mathscr{B}_{\mathbb{R}^n}$ and let $\mathscr{D}(E) = \{E + x | x \in \mathbb{R}^n\}$. Sapogov has shown that $\mathscr{D}(E)$ is a determining class for the family

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of all probability measures on $\mathbb{R}^n$ if either E has positive finite Lebesgue measure or the support of the Fourier-transform of χ_E contains an open subset of $\mathbb{R}^n$.

This paper is concerned with the following situation: let $(G, \mathscr{B}_G)$ be a measurable group acting on a measurable space $(X, \mathscr{B}_X)$ and let μ, ν be two probability measures on $(X, \mathscr{B}_X)$. Do there exist sets $E \in \mathscr{B}_X$ such that whenever $\mu(g \cdot E) = \nu(g \cdot E)$ for every $g \in G$, then $\mu = \nu$? That is, can the action of the group G on X be utilized to find determining classes for probability measures on $(X, \mathscr{B}_X)$? Let a set $E \in \mathscr{B}_X$ be called a *G-determining set* for a family $\mathscr{P}$ of probability measures on $(X, \mathscr{B}_X)$ if the class $\{g \cdot E | g \in G\}$ is a determining class for $\mathscr{P}$. The aim of this paper is to find G-determing sets and to analyse the size of the class of all determining sets using the Baire Category Theorem appropriately.

§1. Definitions and Examples

Let $(X, \mathscr{B}_X)$ be a measurable space. Let $(G, \mathscr{B}_G)$ be a *measurable group*, i.e., G is a group and $\mathscr{B}_G$ is a σ-algebra of subsets of G such that the map

$$(g, h) \to g h^{-1}, \quad g, h \in G$$

is a measurable map from $(G \times G, \mathscr{B}_G \times \mathscr{B}_G)$ into $(G, \mathscr{B}_G)$. By an *action* of $(G, \mathscr{B}_G)$ on $(X, \mathscr{B}_X)$ we mean a measurable map

$$\phi: G \times X \to X$$

such that the following holds:

(i) $\phi(e, x) = x$ for every $x \in X$, e being the identity element of the group G;
(ii) $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$ for every $g_1, g_2 \in G$ and $x \in X$;
(iii) For every $g \in G$, the map

$$x \to \phi(g, x), \quad x \in X$$

is a measurable map from X into itself.

Whenever such a map ϕ exists, we say that G *acts* on X. We shall sometimes denote $\phi(g, x)$ by $g \cdot x$ or just $g x$. For $g \in G$ and $E \in \mathscr{B}_G$, we shall write $g \cdot E$ or just $g E$ for the set $\{g \cdot x | x \in E\}$.

1.1. *Definition.* Let $(X, \mathscr{B}_X)$ be a measurable space and let a measurable group $(G, \mathscr{B}_G)$ act on it. Let ϕ be the action. A set $E \in \mathscr{B}_X$ is called a (G, ϕ)-*determining set* for a family $\mathscr{P}$ of probability measures on $(X, \mathscr{B}_X)$ if E has the following property: for $\mu \nu \in \mathscr{P}$, if

$$\mu(g \cdot E) = \nu(g \cdot E) \quad \text{for every} \quad g \in G,$$

then $\mu = \nu$.

Whenever, it is clear from the context what G and ϕ are, we shall call a (G, ϕ)-determining set simply a determining set.

1.2. *Determining Sets in Euclidean Spaces.* Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and let $\mathscr{B}_{\mathbb{R}^n}$ denote the σ-algebra of Borel subsets of $\mathbb{R}^n$. Let $\lambda_{\mathbb{R}^n}$

denote the Lebesgue measure on $\mathbb{R}^n$ and let $\mathcal{M}(\mathbb{R}^n)$ denote the set of all probability measures on $\mathbb{R}^n$.

1.3. **Lemma.** *Let $E_i \in \mathcal{B}_{\mathbb{R}^n}$, $1 \leqq i \leqq k$, be bounded sets such that $\lambda_{\mathbb{R}^n}\left(\bigcup_{i=1}^{k} E_i\right) > 0$. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$ and let $\sum_{i=1}^{k} \mu(x+E_i) = \sum_{i=1}^{k} \nu(x+E_i)$ for every $x \in \mathbb{R}^n$. Then $\mu = \nu$. In particular, every bounded set $E \in \mathcal{B}_{\mathbb{R}^n}$ with $\lambda_{\mathbb{R}^n}(E) > 0$ is a determining set for $\mathcal{M}(\mathbb{R}^n)$.*

Proof. When $k=1$, the proof is given in Parthasarathy [8] p. 281. The same proof goes through for any k.

Let $\mathbb{R}_+$ denote the multiplicative group of positive real numbers. Let $\mathbb{R}_+^n$ denote the n-fold cartesian product of $\mathbb{R}_+$ with itself. Let $\mathcal{B}_{\mathbb{R}_+^n}$ denote the σ-algebra of Borel subsets of $\mathbb{R}_+^n$ and let $\lambda_{\mathbb{R}_+^n}$ denote the Haar measure of $\mathbb{R}_+^n$. Let $D(n, \mathbb{R}_+)$ denote the group of all $n \times n$ diagonal non-singular matrices with entries from $\mathbb{R}_+$.

1.4. **Lemma.** *Let $D(n, \mathbb{R}_+)$ act on $\mathbb{R}_+^n$ in the natural way. Let $E_i \in \mathcal{B}_{\mathbb{R}_+^n}$, $1 \leqq i \leqq k$ be bounded sets with $\lambda_{\mathbb{R}_+^n}\left(\bigcup_{i=1}^{k} E_i\right) > 0$. Let μ and ν be probability measures on $\mathbb{R}_+^n$ such that $\sum_{i=1}^{k} \mu(g \cdot E_i) = \sum_{i=1}^{k} \nu(g \cdot E_i)$ for every $g \in D(n, \mathbb{R}_+)$. Then $\mu = \nu$.*

Proof. Define the map Exp: $\mathbb{R}^n \to \mathbb{R}_+^n$ as follows:

$$\operatorname{Exp}(x_1, \ldots, x_n) = (e^{x_1}, \ldots, e^{x_n}),\ (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then, Exp is an isomorphism of $\mathbb{R}^n$ onto $\mathbb{R}_+^n$. The inverse map, Log: $\mathbb{R}_+^n \to \mathbb{R}^n$ is given by

$$\operatorname{Log}(t_1, \ldots, t_n) = (\log t_1, \ldots, \log t_n),\ (t_1, \ldots, t_n) \in \mathbb{R}_+^n.$$

Let $\tilde{\mu} = \mu \operatorname{Exp}$ and $\tilde{\nu} = \nu \operatorname{Exp}$. Then $\tilde{\mu}$ and $\tilde{\nu}$ are probability measures on $\mathbb{R}^n$ and from the given conditions on μ and ν it follows that

$$\sum_{i=1}^{k} \tilde{\mu}(x + \operatorname{Log}(E_i)) = \sum_{i=1}^{k} \tilde{\nu}(x + \operatorname{Log}(E_i)), \qquad \text{for every } x \in \mathbb{R}^n.$$

Clearly, each $\operatorname{Log}(E_i)$, $1 \leqq i \leqq k$, is a bounded subset of $\mathbb{R}^n$ and $\lambda_{\mathbb{R}^n}\left(\bigcup_{i=1}^{k} \operatorname{Log}(E_i)\right) > 0$. Thus, it follows from Lemma 1.3, that $\tilde{\mu} = \tilde{\nu}$ and hence $\mu = \nu$.

A probability measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ is called *non-atomic* if $\mu(S) = 0$ for every subspace S of $\mathbb{R}^n$ of dimension less than or equal to $n-1$. For $\mu \in \mathcal{M}(\mathbb{R}^n)$, let $\bar{\mu}$ denote the probability measure on $\mathbb{R}^n$ defined by, $\bar{\mu}(A) = \mu(-A)$, $A \in \mathcal{B}_{\mathbb{R}^n}$. $\mu \in \mathcal{M}(\mathbb{R}^n)$ is said to be symmetric if $\mu = \bar{\mu}$.

Let $U(n, \mathbb{R})$ denote the group of all $n \times n$ non-singular uppertraingular matrices with entries from $\mathbb{R}$.

1.5. **Theorem.** *Let $U(n, \mathbb{R})$ act on $\mathbb{R}^n$ in the natural way. Let $E \in \mathcal{B}_{\mathbb{R}^n}$ be a bounded set with $\lambda_{\mathbb{R}^n}(E) > 0$. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$ be non-atomic probability measures such that*

$\mu(g\cdot E)=\nu(g\cdot E)$ *for every* $g\in U(n,\mathbb{R})$. *Then* $\mu+\bar{\mu}=\nu+\bar{\nu}$. *In particular,* E *is a* $U(n,\mathbb{R})$*-determining set for the family of all non-atomic symmetric probability measures on* $\mathbb{R}^n$.

Proof. To avoid the notational difficulties, we shall give the proof for the case when $n=2$. The proof for general n can be given on the similar lines.

For $\eta\in\mathcal{M}(\mathbb{R}^2)$ and $g\in U(2,\mathbb{R})$, we shall denote by η_g the probability measure on $\mathbb{R}^2$ defined by

$$\eta_g(A)=\eta(g\cdot A),\qquad A\in\mathcal{B}_{\mathbb{R}^2}.$$

Let $g_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $g_2=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$, $g_3=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ and $g_4=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Let $g_0\in U(2,\mathbb{R})$ be chosen arbitrarily and fixed. Define $\tilde{\mu}$ and $\tilde{\nu}$ on $\mathbb{R}^2$ as follows: for $A\in\mathcal{B}_{\mathbb{R}^2}$

$$\tilde{\mu}(A)=\sum_{i=1}^{4}\mu_{g_0g_i}(A),$$

$$\tilde{\nu}(A)=\sum_{i=1}^{4}\nu_{g_0g_i}(A).$$

Claim. $\tilde{\mu}=\tilde{\nu}$.

First note that $\tilde{\mu}$ and $\tilde{\nu}$ are non-atomic probability measures on $\mathbb{R}^2$ and both are invariant under g_i, $1\leqq i\leqq 4$. Thus to prove the claim, it is sufficient to show that $\tilde{\mu}=\tilde{\nu}$ on $\mathbb{R}^2_+$. Let

$$E_1=\{(x,y)\in E\mid x>0,y>0\},$$
$$E_2=\{(x,y)\in E\mid x<0,y>0\},$$
$$E_3=\{(x,y)\in E\mid x<0,y<0\},$$
$$E_4=\{(x,y)\in E\mid x>0,y<0\}$$

and let $A_i=g_i\cdot E_i$, $1\leqq i\leqq 4$.

Then, $A_i\subseteq\mathbb{R}^2_+$ is a bounded set for every i, $1\leqq i\leqq 4$, and $\lambda_{\mathbb{R}^2}\left(\bigcup_{i=1}^{4}A_i\right)>0$.

Further, from the given conditions on μ and ν, it is easy to check that for every $g\in D(2,\mathbb{R}_+)$,

$$\sum_{i=1}^{4}\tilde{\mu}(g\cdot A_i)=\sum_{i=1}^{4}\tilde{\nu}(g\cdot A_i).$$

Thus, it follows from Lemma 1.4 that $\tilde{\mu}=\tilde{\nu}$ on $\mathbb{R}^2_+$, and hence the claim.

Since $g_0\in U(n,\mathbb{R})$ was chosen arbitrarily, we get

$$\sum_{i=1}^{4}\mu_{gg_i}=\sum_{i=1}^{4}\nu_{gg_i}\qquad\text{for every } g\in U(n,\mathbb{R}).$$

Taking the Fourier-transform of both sides, we get for $(t_1,t_2)\in\mathbb{R}^2$ and $g\in U(n,\mathbb{R})$

$$\int_{\mathbb{R}^2}\left[e^{-i(t_1x_1+t_2x_2)}+e^{i(t_1x_1-t_2x_2)}+e^{-i(t_1x_1-t_2x_2)}+e^{i(t_1x_1+t_2x_2)}\right]\mu_g\,d(x_1,x_2)$$
$$=\int_{\mathbb{R}^2}\left[e^{-i(t_1x_1+t_2x_2)}+e^{i(t_1x_1-t_2x_2)}+e^{-i(t_1x_1-t_2x_2)}+e^{i(t_1x_1+t_2x_2)}\right]\nu_g\,d(x_1,x_2)$$

i.e.,

$$\int_{\mathbb{R}^2} \cos(t_1 x_1)\cdot\cos(t_2 x_2)\,\mu_g\,d(x_1,x_2)=\int_{\mathbb{R}^2}\cos(t_1 x_1)\cdot\cos(t_2 x_2)\,\nu_g\,d(x_1,x_2), \quad (1.1)$$

for every $t_1,t_2\in\mathbb{R}$ and $g\in U(n,\mathbb{R})$. Let for $g\in U(n,\mathbb{R})$, $g^{-1}=\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}$, $a,b,c\in\mathbb{R}$ such that $ac\neq 0$. Then by change of variable Equation (1.1) can be written as

$$\int_{\mathbb{R}^2}\cos[t_1(a x_1+b x_2)]\cdot\cos(t_2 x_2 c)\,\mu\,d(x_1,x_2)$$

$$=\int_{\mathbb{R}^2}\cos[t_1(a x_1+b x_2)]\cdot\cos(t_2 x_2 c)\,\nu\,d(x_1,x_2)$$

for every $t_1,t_2,a,b,c\in\mathbb{R}$ such that $ac\neq 0$. In particular if we take $t_1=1$, $t_2=0$ and $t_1=0$, $t_2=1$, we get

$$\int_{\mathbb{R}^2}\cos(a x_1+b x_2)\,\mu\,d(x_1,x_2)=\int_{\mathbb{R}^2}\cos(a x_1+b x_2)\,\nu\,d(x_1,x_2)$$

for every $a,b\in\mathbb{R}$, $a\neq 0$ and

$$\int_{\mathbb{R}^2}\cos(c x_2)\,\mu\,d(x_1,x_2)=\int_{\mathbb{R}^2}\cos(c x_2)\,\nu\,d(x_1,x_2)$$

for every $c\in\mathbb{R}$, $c\neq 0$. Equivalently

$$\text{Real part }[\hat{\mu}(a,b)]=\text{Real part }[\hat{\nu}(a,b)]$$

for every $a,b\in\mathbb{R}$.

Hence $\mu+\bar{\mu}=\nu+\bar{\nu}$.

1.6. *Remark.* The conclusion of Theorem 1.5 cannot be improved even if the group $U(n,\mathbb{R})$ is enlarged to the group $GL(n,\mathbb{R})$, the group of all $n\times n$ non-singular real matrices. To see this, let μ and ν be probability measures on $\mathbb{R}^n$ such that $\mu+\bar{\mu}=\nu+\bar{\nu}$, but $\mu\neq\nu$. Let $E\in\mathcal{B}_{\mathbb{R}^n}$ be such that $E=-E$. Then for every $g\in GL(n,\mathbb{R})$

$$\begin{aligned}2\mu(g\cdot E)&=\mu(g\cdot E)+\mu(g\cdot E)\\&=\mu(g\cdot E)+\mu(g\cdot(-E))\\&=\mu(g\cdot E)+\bar{\mu}(g\cdot E)\\&=\nu(g\cdot E)+\bar{\nu}(g\cdot E)\\&=2\nu(g\cdot E).\end{aligned}$$

Thus, $\mu(g\cdot E)=\nu(g\cdot E)$ for every $g\in GL(n,\mathbb{R})$ but $\mu\neq\nu$.

§2. Determining Sets in Locally-Compact Second Countable Groups

Let G be a locally compact second countable group and let $\mathcal{B}_G$ be the σ-algebra of Borel subsets of G. Let λ_G be a Haar measure of G and let $\mathcal{M}(G)$ denote the

family of all probability measures on G. For $\mu, \nu \in \mathcal{M}(G)$, let $\mu * \nu$ denote the *convolution* of μ with ν. Whenever, G is abelian also, let $\hat{G}$ denote the character group of G. For $\mu \in \mathcal{M}(G)$, let $\hat{\mu}$ denote the *Fourier-transform* of μ.

Let G act on itself by multiplication on the left. Let $E \in \mathcal{B}_G$ be a set with compact closure and let $\lambda_G(E) > 0$. In view of §1.2, one asks the question: is E a determining set for $\mathcal{M}(G)$? We show by an example that the above question cannot be answered in the affirmative.

Let $G = \mathbb{Z} \times K$, where $\mathbb{Z}$ is the integer group and K is any compact abelian group. Choose two probability measures μ_1 and μ_2 on K such that $\mu_1 \neq \mu_2$. Choose a non-zero probability measure λ on $\mathbb{Z}$. Put $\mu = \lambda \times \mu_1$ and $\nu = \nu \times \mu_2$. Let $E = \{0\} \times K$. Then E is a compact subset of G and $\lambda_G(E) > 0$. Further, it is easy to check that $\mu(g+E) = \nu(g+E)$ for every $g \in G$, but $\mu \neq \nu$. Thus E is not a determining set for $\mathcal{M}(G)$. However, consider $\theta = \delta_0 \times \lambda_K$, where δ_0 denotes the probability measure degenerate at $0 \in \mathbb{Z}$ and λ_K denotes the normalised Haar measure of K. It is easy to see that $\mu * \theta = \nu * \theta$.

The situation for a general group is described by the following:

2.1. **Theorem.** *Let G be a locally-compact second countable abelian group and let $E \in \mathcal{B}_G$ be such that $\lambda_G(E) > 0$ and the closure of E is compact. Let G_0 be the subgroup of G generated by E and let K be the maximal compact subgroup of G_0. Let $\mu, \nu \in \mathcal{M}(G)$ be such that $\mu(g+E) = \nu(g+E)$ for every $g \in G$. Then $\mu * \lambda_K = \nu * \lambda_K$, λ_K being the normalised Haar measure of K. In particular, E is a determining set for the family of all K-invariant probability measures on G.*

Proof. Refer Rana [9].

2.2. **Lemma.** *Let G be a locally-compact second countable abelian group. Let $E \in \mathcal{B}_G$ be such that $0 < \lambda_G(E) < \infty$ and such that $\{\gamma \in \hat{G} \mid \hat{\chi}_E(\gamma) \neq 0\}$ is a dense subset of $\hat{G}$. Then E is a determining set for $\mathcal{M}(G)$.*

Proof. Let $\mu, \nu \in \mathcal{M}(G)$ be such that

$$\mu(g+E) = \nu(g+E) \quad \text{for every } g \in G. \tag{2.1}$$

We have to show that $\mu = \nu$. Condition (2.1) is equivalent to

$$\int \chi_E(g+h)\, \mu(dh) = \int \chi_E(g+h)\, \nu(dh), \quad \text{for every } g \in G.$$

Thus for every $\gamma \in \hat{G}$

$$\int \langle \overline{g, \gamma} \rangle \left(\int \chi_E(g+h)\, \mu(dh) \right) \lambda_G(dg) = \int \langle \overline{g, \gamma} \rangle \left(\int \chi_E(g+h)\, \nu(dh) \right) \lambda_G(dg).$$

Using Fubini's theorem and changing g to $g-h$, we get

$$\int \left(\int \chi_E(g) \langle \overline{g-h, \gamma} \rangle\, \lambda_G(dg) \right) \mu(dh) = \int \left(\int \chi_E(g) \langle \overline{g-h, \gamma} \rangle\, \lambda_G(dg) \right) \nu(dh).$$

Since each $\gamma \in \hat{G}$ is a homomorphism, we have

$$\int \left(\int \chi_E(g) \langle \overline{g, \gamma} \rangle \langle \overline{-h, \gamma} \rangle\, \lambda_G(dg) \right) \mu(dh) = \int \left(\int \chi_E(g) \langle \overline{g, \gamma} \rangle \langle \overline{-h, \gamma} \rangle\, \lambda_G(dg) \right) \nu(dh).$$

Another application of Fubini's theorem gives us

$$\begin{aligned}&\left(\int \chi_E(g)\overline{\langle g,\gamma\rangle}\,\lambda_G(dg)\right)\left(\int \overline{\langle -h,\gamma\rangle}\,\mu(dh)\right)\\&=\left(\int \chi_E(g)\overline{\langle g,\gamma\rangle}\,\lambda_G(dg)\right)\left(\int \overline{\langle -h,\gamma\rangle}\,\nu(dh)\right)\end{aligned}\tag{2.2}$$

for every $\gamma\in\hat{G}$. For $\eta\in\mathscr{M}(G)$, let $\bar{\eta}$ denote the probability measure on G defined by

$$\bar{\eta}(A)=\eta(-A),\qquad A\in\mathscr{B}_G,$$

where $-A=\{-g|g\in A\}$. Then Equation (2.2) can be written as

$$\hat{\chi}_E(\gamma)\hat{\bar{\mu}}(\gamma)=\hat{\chi}_E(\gamma)\hat{\bar{\nu}}(\gamma)\qquad\text{for every } \gamma\in\hat{G}.\tag{2.3}$$

Since, the set $\{\gamma\in\hat{G}|\hat{\chi}_E(\gamma)\neq 0\}$ is dense in $\hat{G}$ and $\hat{\bar{\mu}},\hat{\bar{\nu}}$ are continuous functions on $\hat{G}$, Equation (2.3) implies that

$$\hat{\bar{\mu}}(\gamma)=\hat{\bar{\nu}}(\gamma)\qquad\text{for every } \gamma\in\hat{G}.$$

Hence $\mu=\nu$.

2.3. **Corollary.** *Let $\mathbb{T}$ denote the circle group, i.e., the multiplicative group of all complex numbers of absolute value one. Let $E=\{e^{i\theta}\in\mathbb{T}|0\leqq\theta<\delta\}$, where δ is an irrational multiple of π. Then E is a determining set for the family of all probability measures on $\mathbb{T}$.*

Proof. In view of Lemma 2.2, we have only to show that the set $\{\gamma\in\hat{\mathbb{T}}|\hat{\chi}_E(\gamma)\neq 0\}$ is a dense subset of $\hat{\mathbb{T}}$. Let $m\in\hat{\mathbb{T}}=\mathbb{Z}$, the integer group. Then

$$\hat{\chi}_E(m)=\frac{1}{2\pi}\,\frac{e^{-im\delta}-1}{-im}.$$

Since, δ is an irrational multiple of π, it follows that $\hat{\chi}_E(m)\neq 0$ for every $m\in\mathbb{Z}$.

2.4. *Remark (1).* The condition that δ is an irrational multiple of π is necessary. To see this, we construct probability measures μ and ν on $\mathbb{T}$ such that

(i) μ is singular to ν;
(ii) μ and ν agree on all arcs of length π. (2.4)

Without loss of generality, we may assume that $\mathbb{T}$ is the circle with center having Cartesian co-ordinates $\{0,1\}$ and with unit radius. Then using the stereographic-projection of the circle onto the real-line $\mathbb{R}$, it is easy to see that constructing probability measures μ and ν on $\mathbb{T}$ satisfying (2.4) is equivalent to constructing probability measures $\tilde{\mu}$ and $\tilde{\nu}$ on $\mathbb{R}$ satisfying the following:

(i) $\tilde{\mu}$ is singular to $\tilde{\nu}$;
(ii) $\tilde{\mu}\left(-x,\frac{1}{x}\right)=\tilde{\nu}\left(-x,\frac{1}{x}\right)$ for every $x>0$. (2.5)

Probability measures $\tilde{\mu}$ and $\tilde{\nu}$ on $\mathbb{R}$ satisfying (2.5) can be defined as follows: for any $A\subseteq\mathbb{R}\setminus\{0\}$, let $-A^{-1}=\left\{-\frac{1}{x}\middle| x\in A\right\}$. Choose a probability measure λ on the

interval $(0,1)$ and a probability measure η on the interval $(-1,0)$. For $A\in\mathscr{B}_{\mathbf{R}}$, define

$$\begin{aligned}\tilde{\mu}(A)&=\tfrac{1}{2}\lambda(-A^{-1}) && \text{if } A\subseteq(-\infty,-1);\\ &=0 && \text{if } A\subseteq[-1,0]\cup[1,\infty);\\ &=\tfrac{1}{2}\lambda(A) && \text{if } A\subseteq(0,1)\end{aligned}$$

and

$$\begin{aligned}\tilde{\nu}(A)&=0 && \text{if } A\subseteq(-\infty,-1]\cup[0,1];\\ &=\tfrac{1}{2}\eta(A) && \text{if } A\subseteq(-1,0);\\ &=\tfrac{1}{2}\eta(-A^{-1}) && \text{if } A\subseteq(1,\infty).\end{aligned}$$

Let G be a locally-compact second countable abelian group. We showed in Lemma 2.2, that every $E\in\mathscr{B}_G$ with $0<\lambda_G(E)<\infty$ and such that the set $\{\gamma\in\hat{G}\mid\hat{\chi}_E(\gamma)\neq 0\}$ is dense in $\hat{G}$, is a determining set for $\mathscr{M}(G)$. We also exhibited that not every set $E\in\mathscr{B}_G$ with $0<\lambda_G(E)<\infty$ is a determining set for $\mathscr{M}(G)$. This leads to the natural question: how big is the class of all determining sets for $\mathscr{M}(G)$? We shall show that generically a set of positive finite Haar measure in G is a determining set for $\mathscr{M}(G)$. Let

$$s(G)=\{E\in\mathscr{B}_G\mid\lambda_G(E)<\infty\}.$$

Let $A, B\in s(G)$. We say that A is equivalent to B whenever $\lambda_G(A\,\Delta\,B)=0$, Δ being the symmetric difference. This is an equivalence relation on $s(G)$. Let the set of equivalence of $s(G)$ under this relation be denoted by $s(G)$ itself. For $A, B\in s(G)$, define

$$d(A,B)=\lambda_G(A\,\Delta\,B).$$

Then d is a well defined metric on $s(G)$ and $(s(G),d)$ is a complete metric space. For every real number $\alpha>0$, let

$$s_\alpha(G)=\{E\in s(G)\mid\lambda_G(E)<\alpha\}.$$

Then $s_\alpha(G)$ is an open subset of $(s(G),d)$. Let d_α denote the restriction of the metric d to $s_\alpha(G)$. Let d'_α denote a metric on $s_\alpha(G)$ such that d'_α is equivalent to d_α and such that $(s_\alpha(G),d'_\alpha)$ is a complete metric space, (that such a metric always exists, see Dieudonne [6] p. 55).

2.5. **Lemma.** *Let G be a locally-compact second countable group and let $f\colon G\to\mathbb{C}$ be a bounded continuous function such that $f(e)=1$. Then the set*

$$D(\alpha;f)=\{E\in s_\alpha(G)\mid\int_E f(g)\,\lambda_G(dg)\neq 0\}$$

is open and dense in $(s_\alpha(G),d'_\alpha)$.

Proof. Clearly, $D(\alpha;f)$ is open in $s_\alpha(G)$. To show that $D(\alpha;f)$ is dense in $s_\alpha(G)$, we have to show that for every $\delta>0$ and $E\in s_\alpha(G)$, there exists an $A\in D(\alpha;f)$ such that $\lambda_G(A\,\Delta\,E)<\delta$.

Let $E\in s_\alpha(G)$ be chosen arbitrarily and fixed. Let $\delta>0$ be arbitrary. Without loss of generality, let $\delta<1$. If $E\in D(\alpha;f)$, then there is nothing to prove. So, let $E\notin D(\alpha;f)$, i.e., $\int_E f(g)\,\lambda_G(dg)=0$.

Since f is continuous and $f(e)=1$, we can choose an open neighbourhood N of $e\in G$ such that $0<\lambda_G(N)<\min(\delta, \alpha-\lambda_G(E))$ and $|1-f(g)|<\delta$ for every $g\in N$. Then

$$\begin{aligned}\left|\int_N f(g)\,\lambda_G(dg)\right| &\geqq \lambda_G(N)-\int_N |1-f(g)|\,\lambda_G(dg)\\ &>(1-\delta)\,\lambda_G(N)\\ &>0.\end{aligned}$$

Thus, $\int_N f(g)\,\lambda_G(dg)\neq 0$. Now two cases can arise:

Case (1). $\int_{E\cup N} f(g)\,\lambda_G(dg)\neq 0$.

Put $A=E\cup N$. Then $\int_A f(g)\,\lambda_G(dg)\neq 0$ and

$$\begin{aligned}0<\lambda_G(A)&=\lambda_G(E\cup N)\\ &\leqq \lambda_G(E)+\lambda_G(N)\\ &<\alpha.\end{aligned}$$

Thus $A\in D(\alpha; f)$ and clearly $\lambda_G(A\,\triangle\, E)<\lambda_G(N)<\delta$.

Case (ii). $\int_{E\cup N} f(g)\,\lambda_G(dg)=0$.

Put $A=E\setminus(E\cap N)$. Then

$$\int_A f(g)\,\lambda_G(dg)+\int_N f(g)\,\lambda_G(dg)=\int_{E\cup N} f(g)\,\lambda_G(dg)=0$$

i.e.,

$$\int_A f(g)\,\lambda_G(dg)=-\int_N f(g)\,\lambda_G(dg)\neq 0.$$

Further, clearly $0<\lambda_G(A)<\alpha$ and $\lambda_G(A\,\triangle\, E)<\delta$.

Thus, in either case there exists an $A\in D(\alpha; f)$ such that $\lambda_G(A\,\triangle\, E)<\delta$.

2.6. **Theorem.** *Let G be a locally-compact second countable abelian group. Then for every real number $\alpha>0$, there exists $D(\alpha; G)\subset \mathscr{B}_G$ such that the following hold:*

(i) *$D(\alpha; G)$ is a dense G_δ-subset of $(s_\alpha(G), d'_\alpha)$;*

(ii) *Every $E\in D(\alpha; G)$ is a determining set for $\mathscr{M}(G)$.*

Proof. Since G is second countable, its dual group $\hat{G}$ is separable. Let $\gamma_1, \gamma_2, \cdots$ be a dense subset of $\hat{G}$. For each j, γ_j is a bounded complex-valued function on G and $\gamma_j(e)=1$. Thus, it follows from Lemma 2.5 that the set

$$D(\alpha; \gamma_j)=\{E\in s_\alpha(G) \mid \int_E \overline{\langle g, \gamma_j\rangle}\,\lambda_G(dg)\neq 0\}$$

is an open dense subset of $(s_\alpha(G), d'_\alpha)$. Let

$$D(\alpha; G)=\bigcap_{j=1}^{\infty} D(\alpha; \gamma_j).$$

Since $(s_\alpha(G), d'_\alpha)$ is a complete metric space, by Bair's Category Theorem it follows that $D(\alpha; G)$ is a dense G_δ-subset of $(s_\alpha(G), d'_\alpha)$.

Let $E \in D(\alpha; G)$. Then $0 < \lambda_G(E) < \alpha$ and the set $\{\gamma \in \hat{G} | \hat{\chi}_E(\gamma) \neq 0\}$ is a dense subset of $\hat{G}$, since it contains γ_j for every $j = 1, 2, \ldots$. Thus by Lemma 2.2, E is a determining set for $\mathcal{M}(G)$.

2.7. *Remark (1).* Let G be a locally-compact second countable (not necessarily) abelian group. Let G act on itself by left-multiplication. The following questions remain unanswered:

(i) Does there exist a determining set for $\mathcal{M}(G)$?

(ii) Let $E \in \mathcal{B}_G$ be such that $0 < \lambda_G(E) < \infty$. Can one associate a compact subgroup K to E such that E becomes a determining set for the family of all right (left) K-invariant probability measures on G?

2.7. *Remark (2).* Let G be a locally-compact second countable group. Let K be a compact open subgroup of G. Let $\mu, \nu \in \mathcal{M}(G)$ be such that $\mu(gK) = \nu(gK)$ for every $g \in G$. Then $\mu * \lambda_K = \nu * \lambda_K$, λ_K being the normalised Haar measure of K. In particular K is a determining set for the family of all right K-invariant probability measures on G.

To see this, first note that for every $g \in G$, the probability measures $\delta_{g^{-1}} * \mu * \lambda_K$ and $\delta_{g^{-1}} * \nu * \lambda_K$ are both right K-invariant and $(\delta_{g^{-1}} * \mu * \lambda_K)(K) = (\delta_{g^{-1}} * \nu * \lambda_K)(K)$. Thus $\delta_{g^{-1}} * \mu * \lambda_K = \delta_{g^{-1}} * \nu * \lambda_K$ on K. Equivalently, $\mu * \lambda_K = \nu * \lambda_K$ on every left-coset gK, $g \in G$. Since K is open and G is second countable, there are only countably many cosets.

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A Martingale Method for the Convergence of a Sequence of Processes to a Jump-diffusion Process

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Summary. A convenient method for proving weak convergence of a sequence of non-Markovian processes $x^\varepsilon(\cdot)$ to a jump-diffusion process is proved. Basically, it is shown that the limit solves the martingale problem of Strook and Varadhan. The proofs are relatively simple, and the conditions apparently weaker than required by other current methods (in particular, for limit theorems for a sequence of ordinary differential equations with random right hand sides). In order to illustrate the relative ease of applicability in many cases, a simpler proof of a known result on averaging is given.

1. Introduction

In [6–8], Kurtz developed a general theory for the weak convergence of a sequence $\{x^\varepsilon(\cdot)\}$ of (not necessarily Markov) processes to a Markov process $x(\cdot)$. The usefulness of his approach is attested to by the applications in [4, 9]. Kurtz's method involved a general approach to the problem of semigroup approximation and using it in [4] we were able to improve upon the approximations results of (e.g.) [3, 10], with both simpler proofs and better conditions. Reference [4] dealt with the weak convergence of a sequence of solutions to ordinary differential equations with random right hand sides to jump-diffusion process.

In this paper, some of the remarks and ideas on proving tightness in [8] are exploited to get a result of the same type in a simpler way. The general theory of [8] is not used. Instead of showing weak convergence to a Markov process, we show weak convergence to a solution of the martingale problem of Strook and Varadhan [12–13]. Our approach has several advantages: the proofs are short and straightforward, less care is required in the choice of the test functions

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(Kurtz's dense class $\mathscr{D}$), the conditions are often easier to verify—especially for the non-homogeneous case (e.g., we do not need to deal with the density of $(A-\lambda)\mathscr{D}$, see [8]). Finally, the extension to convergence of continuous parameter interpolations of discrete parameter processes is easier and more natural in that we do not need to treat the interpolation as a continuous parameter process—but only look at it at the "jump points". Also, for the discrete parameter case, our method often allows the perturbations[1] $\{f^\varepsilon\}$ of the test function f to be obtained by relatively straightforward means. Some additional remarks on this case appear in Sect. 4, but a fuller exposition of the discrete parameter will be given in a subsequent paper, together with several applications.

The problem is set up in Sect. 2 and the main theorem proved in Sect. 3. Some remarks on tightness are made in Sect. 3, the discrete parameter case is treated in Sect. 4 and Sect. 5 illustrates the simplicity of the idea by giving a simpler proof of a classical problem of averaging. One possible disadvantage of our approach is that tightness must be proved first. This does not seem to be a handicap however, since in many applications, one can do this quite easily by the approximation method developed in the sequel and the tightness results in Kurtz's paper [8]. This is illustrated in Theorems 2, 5, or by the way tightness is proved in [4, 5], where Kurtz's scheme [8] is used.

Since writing this paper, it has come to the author's attention that Papanicolaou, Strook and Varadhan [11] have used a related method to get a variety of limit theorems. That work is restricted to a purely continuous parameter and Markovian framework and, except for some special results which do not overlap ours, the Markov driving noise (analogous to $y(\cdot)$ in (3.6a)) is required to have a strong ergodic property. Within their ergodic-Markov framework, they treat a wider class of problems than we do here. Our $x^\varepsilon(\cdot)$ are not necessarily Markovian, and various ideas of Kurtz [8] are adopted to prove the requisite tightness.

2. Problem Formulation

Let $\{x^\varepsilon(\cdot)\}$ denote a sequence of R^r valued processes with paths in $D^r[0,\infty)$ (endowed with the Skorohod topology), and all (w.l.o.g.) defined on the same sample space $(\Omega, P, \mathscr{F})$. Let $\{\mathscr{F}_t\}$ denote an increasing sequence of sub σ-algebras of $\mathscr{F}$ such that $\{x^\varepsilon(s), s \leqq t\}$ is $\mathscr{F}_t$ measurable. Let $\mathscr{L}$ denote the progressively measurable functions with respect to $\{\mathscr{F}_t\}$. There are progressively measurable versions of all the functions introduced below, and we assume that those are the versions used. Define $\bar{\mathscr{L}}$ to be the subset of $\mathscr{L}$ for which $\sup_t E|f(t)| < \infty$. Let $S_N = \{x: |x| \leqq N\}$. Let $\mathscr{F}_t^\varepsilon \subset \mathscr{F}_t$ denote an increasing sequence of σ-algebras which measures $x^\varepsilon(s)$, $s \leqq t$, and let E_t^ε denote expectation conditioned on $\mathscr{F}_t^\varepsilon$. Define the operators p-lim and $\hat{A}^\varepsilon$ as in [8] by: For f and $f^\delta \in \bar{\mathscr{L}}$, we say that $p\text{-}\lim_{\delta\to 0} f^\delta = f$ iff (2.1) holds. (2.2) defines $\hat{A}^\varepsilon$ and $\mathscr{D}(\hat{A}^\varepsilon)$.

$$\sup_{t,\delta} E|f^\delta(t)| < \infty, \quad \lim_{\delta\to 0} E|f^\delta(t) - f(t)| = 0 \quad \textit{each } t; \tag{2.1}$$

[1] We use $\varepsilon \to 0$ rather than $n \to \infty$ (as in [8]) to index the sequence

$$f \in \mathscr{D}(\hat{A}^\varepsilon) \quad \textit{and} \quad \hat{A}^\varepsilon f = g \ \textit{iff for } f, g \in \bar{\mathscr{P}} \textit{ and adapted to } \{\mathscr{F}_t^\varepsilon\} \textit{ and } g \textit{ p-right continuous}^2, \tag{2.2}$$

$$p\text{-}\lim_{\delta \to 0} \left[\frac{1}{\delta}(E_t^\varepsilon f(t+\delta) - f(t)) - g(t)\right] = 0.$$

Let $\hat{\mathscr{C}}$ denote the space of continuous bounded real-valued functions on $R^r \times R^+$ which are zero at infinity, $\hat{\mathscr{C}}_0$ the subset with compact support, and $\hat{\mathscr{C}}_0^{\alpha,\beta}$ (resp., $\hat{\mathscr{C}}^{\alpha,\beta}$) the subset of $\hat{\mathscr{C}}_0$ (resp., $\hat{\mathscr{C}}$) with bounded and continuous α t-partial derivatives and β x-partial derivatives. $\mathscr{B}(R^q)$ denotes the Borel field over R^q.

Let $\tilde{A}$ denote the diffusion operator $\sum_i b_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j}$ where we assume that the coefficients are continuous, let $\lambda(\cdot,\cdot)$ denote a non-negative real-valued bounded continuous function on $R^q \times R^+$. Let $\mu(\cdot,\cdot,\cdot)$ be a function on $R^r \times R^+ \times \mathscr{B}(R^q)$ such that $\mu(x,t,\cdot)$ is a measure of total mass unity on $\mathscr{B}(R^q)$ for each x,t, and such that for each $f \in \hat{\mathscr{C}}$, $\int f(x+y,t)\,\mu(x,t,dy)$ is continuous in x,t.

Let A denote the operator on $\hat{\mathscr{C}}_0^{1,2}$ given by

$$Af(x,t) = \tilde{A}f(x,t) + \lambda(x,t)\int [f(x+y,t) - f(x,t)]\,\mu(x,t,dy).$$

For each x let there be a measure P_x on $D^r[0,\infty)$ such that $P_x\{x(0)=x\}=1$, and

$$\textit{for each } T < \infty, P_x\{\sup_{t \le T} |x(t)| < \infty\} = 1, \tag{2.3}$$

and which is assumed to be the *unique* solution to the martingale problem of Strook and Varadhan [12–13]: vis, for each $f(\cdot) \in \hat{\mathscr{C}}_0^{1,2}$, the function

$$M_f(t) = f(x(t),t) - f(x(0),0) - \int_0^t \left(\frac{\partial}{\partial s} + A\right) f(x(s),s)\,ds \tag{2.4}$$

is a P_x martingale. By the uniqueness, x can be replaced by any random variable X in the above.

3. The Convergence Theorem

Theorem 1 requires tightness; since we do not work within a semigroup framework, we cannot first prove convergence of finite dimensional distributions as done in [8], and then prove tightness. For many problems (as will be seen) this is no barrier to effective use of our approach. It is often much easier to prove tightness when the $\{x^\varepsilon(t)\}$ are bounded uniformly in ε, t. In order to exploit this fact, we introduce a sequence of truncations of $\{x^\varepsilon(\cdot)\}$, prove convergence for each truncation, and then take limits to get convergence for the original sequence. Section 5 illustrates how such an idea can be used for at least one class of problems. See also the remark after the proof of Theorem 1. The truncation

[2] For $g \in \bar{\mathscr{P}}$, p-right continuity means that $\lim_{\delta \downarrow 0} E|g(t+\delta) - g(t)| = 0$ for each $t \ge 0$

method seems quite natural and easy to use for the classes of problems in [3–4]. In fact, the general method for creating the truncations $x^{\varepsilon,N}(\cdot)$ is simply to stop or "slow down" $x^\varepsilon(\cdot)$ when $|x^\varepsilon(t)| \geq N$. The limits of these truncations (as $\varepsilon \to 0$) determine the limit as $\varepsilon \to 0$ of $x^\varepsilon(\cdot)$, under our uniqueness hypothesis. Because of the truncation idea, we introduce the following definition. *Let $X(\cdot)$ denote the process defined by the solution to the martingale problem* (2.4) *with initial condition* $X(0)$ (i.e., with measure $P_{X(0)}$). Let A^N denote an operator of the form of A with coefficients $a^N(\cdot,\cdot)$, $b^N(\cdot,\cdot)$, $\lambda^N(\cdot,\cdot)$, $\mu^N(\cdot,\cdot,\cdot)$ satisfying the conditions in the paragraph above (2.3) and which are equal to the coefficients of A when $|x| \leq N$. Suppose that $X^N(\cdot)$ is a process with paths in $D^r[0,\infty)$ such that $X^N(0) \to X(0)$ weakly and $X^N(\cdot)$ solves (not necessarily uniquely) the martingale problem (2.4) – with initial condition $X^N(0)$. Then $X^N(\cdot)$ is called an N-truncation of $X(\cdot)$.

Theorem 1. *Assume the conditions*[3] *of Sect.* 2. *For each $\varepsilon > 0$, let $\{x^{\varepsilon,N}(\cdot)\}$ be a bounded*[4] *sequence of processes with paths in $D^r[0,\infty)$ satisfying $x^{\varepsilon,N}(t) = x^\varepsilon(t)$ for $t \leq \inf\{s: |x^\varepsilon(s)| > N\}$. Let $x^\varepsilon(0) \to X(0)$ weakly. If $|x^\varepsilon(0)| > N$, set $x^{\varepsilon,N}(t) \equiv 0$. Let $\mathscr{D}$ be a dense (in sup norm) set in $\mathscr{C}$ of functions with compact support, and suppose that $X^N(\cdot)$ is an N-truncation of $X(\cdot)$ (initial condition $X(0)$). Let $\mathscr{F}_t^{\varepsilon,N} \subset \mathscr{F}_t$ be an increasing sequence of σ-algebras measuring $x^{\varepsilon,N}(s)$, $s \leq t$, and define the operator $\hat{A}^{\varepsilon,N}$ (using $\mathscr{F}_t^{\varepsilon,N}$ and $x^{\varepsilon,N}(\cdot)$) just as $\hat{A}^\varepsilon$ was defined (using $\mathscr{F}_t^\varepsilon$ and $x^\varepsilon(\cdot)$). Let $E_t^{\varepsilon,N}$ denote expectation conditioned on $\mathscr{F}_t^{\varepsilon,N}$. Suppose that for each N and each $f(\cdot) \in \mathscr{D}$, there is a sequence $f^{\varepsilon,N}(\cdot) \in \mathscr{D}(\hat{A}^{\varepsilon,N})$ such that*

$$p\text{-}\lim_{\varepsilon \to 0} [f^{\varepsilon,N}(\cdot) - f(x^{\varepsilon,N}(\cdot),\cdot)] = 0 \tag{3.1}$$

$$p\text{-}\lim_{\varepsilon \to 0} [\hat{A}^{\varepsilon,N} f^{\varepsilon,N}(\cdot) - \left(\frac{\partial}{\partial t} + A^N\right) f(x^{\varepsilon,N}(\cdot),\cdot)] = 0. \tag{3.2}$$

Then, if $\{x^{\varepsilon,N}(\cdot)\}$ is tight in $D^r[0,\infty)$ for each N, $\{x^\varepsilon(\cdot)\}$ converges weakly to $X(\cdot)$.

Proof. Fix N and $f(\cdot) \in \mathscr{D}$. Choose $\{f^{\varepsilon,N}(\cdot)\}$ satisfying (3.1)–(3.2). Equation (3.3) follows from the definition of $\hat{A}^{\varepsilon,N}$ (see Kurtz [3])

$$E_t^{\varepsilon,N} f^{\varepsilon,N}(t+s) - f^{\varepsilon,N}(t) = \int_t^{t+s} E_t^{\varepsilon,N} \hat{A}^{\varepsilon,N} f^{\varepsilon,N}(u)\, du. \tag{3.3}$$

Let q denote an arbitrary integer, let $t_i \leq t$, $i \leq q$, and let $g(\cdot)$ be a bounded continuous function on R^{rq}. Then by (3.1)–(3.3),

$$\lim_{\varepsilon \to 0} E g(x^{\varepsilon,N}(t_i), i \leq q) \left[f(x^{\varepsilon,N}(t+s), t+s) - f(x^{\varepsilon,N}(t), t) - \int_t^{t+s} \left(\frac{\partial}{\partial t} + A^N\right) f(x^{\varepsilon,N}(u), u)\, du \right] = 0.$$

---

[3] Recall in particular the most important assumption: that the martingale problem (2.4) has a unique solution on $D^r[0,\infty)$ for each initial condition x

[4] Actually, all that is needed is $\lim_{K \to \infty} \sup_\varepsilon P\{\sup_{t \leq T} |x^{\varepsilon,N}(t)| \geq K\} = 0$ for each $T > 0$, but boundedness does not seem to be a restriction in our truncation method

Choose and fix a weakly convergent subsequence of $\{x^{\varepsilon,N}(\cdot)\}$, indexed also by ε and with limit denoted by $X^N(\cdot)$ and where $X^N(0)=$ weak limit of $\{x^{N,\varepsilon}(0)\}$. Then, by the weak convergence[5] and the continuity properties of the coefficients of A^N,

$$Eg(X^N(t_i), i\leqq q)[M_f^N(t+s)-M_f^N(t)]=0 \tag{3.4}$$

where

$$M_f^N(t)=f(X^N(t),t)-f(X^N(0),0)-\int_0^t\left(\frac{\partial}{\partial t}+A^N\right)f(X^N(u),u)\,du. \tag{3.5}$$

By the density of $\mathscr{D}$ and the arbitrariness of q, $\{t_i\}$ and $g(\cdot)$ and the fact that for each $T<\infty$, $P\{\sup_{t\leqq T}|X^N(t)|<\infty\}=1$, $M_f^N(\cdot)$ is a martingale for each N, $f(\cdot,\cdot)\in\hat{\mathscr{C}}_0^{1,2}$. Thus, there is a measure P_N on $D^r[0,\infty)$ such that under P_N, $x(0)$ has the distribution of $X^N(0)$ and P_N solves (not necessarily uniquely) the martingale problem (2.4) with A^N replacing A. Let $P_{X(0)}$ denote the unique measure solving the martingale problem (2.4) (initial condition $X(0)$). By the uniqueness, $P_{X(0)}$ and P_M must agree on measurable subsets of the set $\{x(\cdot): |x(t)|\leqq N,\ t\leqq T\}\in D^r[0,\infty)$ for each T and $M\geqq N$. From this and (2.3), it follows that $x^\varepsilon(\cdot)\to X(\cdot)$ weakly. Q.E.D.

Remarks. One useful method for getting the $f^{\varepsilon,N}(\cdot)$ from $f(\cdot,\cdot)$ is developed in Sect. 5. A similar method was used in [4, 5], where Kurtz's original method [8] was used to prove weak convergence, and no N-truncations were required. In [4], $\{x^\varepsilon(\cdot)\}$ was defined as follows. For a stationary ϕ-mixing process $y(\cdot)$, define $y^\varepsilon(t)=y(t/\varepsilon^2)$ and for appropriate F, G, let

$$\dot{x}^\varepsilon=G(x^\varepsilon,t,y^\varepsilon)+F(x^\varepsilon,t,y^\varepsilon)/\varepsilon. \tag{3.6a}$$

The form (3.6a) uses a frequently used scaling (see [3, 10]), to get limit theorems for the solutions to a sequence of ordinary differential equations with random right hand sides. Let $q_N(\cdot)$ denote a real valued infinitely differentiable function on R^r which equals one on S_N, is zero on R^r-S_{N+1}, and is bounded by unity. Then, we might define $x^{\varepsilon,N}(\cdot)$ by

$$\dot{x}^{\varepsilon,N}=q_N(x^{\varepsilon,N})[G(x^{\varepsilon,N},t,y^\varepsilon)+F(x^{\varepsilon,N},t,y^\varepsilon)/\varepsilon]. \tag{3.6b}$$

The usefulness of such a truncation will be clear from the examples. Such a truncation device is often useful when the $x^\varepsilon(\cdot)$ is given via an explicit dynamical equation. The method of this paper avoids the need for the density of $(\lambda-A)\mathscr{D}$ in $\mathscr{C}$, for some $\lambda>0$, as required by the method of [8]. Essentially, the density condition is replaced by the uniqueness condition for the solution to the martingale problem.

Tightness. We have assumed, in Theorem 1, that for each N, $x^{\varepsilon,N}(\cdot)$ is bounded uniformly in ε. Thus tightness of $\{x^{\varepsilon,N}(\cdot)\}$ is equivalent to tightness of $\{f(x^{\varepsilon,N}(\cdot))\}$ for each $f(\cdot)\in\hat{C}_0$, the space of continuous bounded real valued functions on R^r with compact support. We prefer to deal with $f(x^{\varepsilon,N}(\cdot))$ rather

[5] For (3.4) to hold, it may be necessary to require that $t_i\notin T_0$, a null set which might depend on $X^N(\cdot)$, but this deletion does not affect the proof

than with $x^{\varepsilon,N}(\cdot)$ directly because, as seen below, it often enables us to use the $f^{\varepsilon,N}(\cdot)$ as done in Theorem 1. The following special case of Kurtz's theorem [8, (4.20)] will be useful.

Lemma 1. *Let $\{y^\varepsilon(\cdot)\}$ be a bounded sequence of processes with paths in $D^r[0,\infty)$. Suppose that for each $T<\infty$, $\delta>0$, there is a random variable $\gamma_\varepsilon(\delta)$ such that*[6]

$$E_t^\varepsilon \gamma_\varepsilon(\delta) \geqq E_t^\varepsilon \min[1, |y^\varepsilon(t+u)-y^\varepsilon(t)|^2], \quad \text{all} \quad u\leqq\delta, t\leqq T, \tag{3.7}$$

and that

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}} E\gamma_\varepsilon(\delta)=0. \tag{3.8}$$

Then $\{y^\varepsilon(\cdot)\}$ is tight in $D^r[0,\infty)$.

For our problem, $y^\varepsilon(\cdot)=f(x^{\varepsilon,N}(\cdot))$, for $f(\cdot)\in\hat{C}_0$. The following useful method of getting the $\gamma_\varepsilon(\delta)$ is suggested by a comment in [8], and is an extension of a result in [4].

Theorem 2. *Fix N. For each $f(\cdot)$ in a dense set $\mathscr{D}_1\subset\hat{C}_0$ which contains the square of each function in it, let there be a sequence $\{f^{\varepsilon,N}(\cdot)\}$ in $\bar{\mathscr{D}}$ such that $f^{\varepsilon,N}(\cdot)\in\mathscr{D}(\hat{A}^{\varepsilon,N})$, and for each real $T>0$ let there be a random variable $M_T^{\varepsilon,N}(f)$ such that*

$$P\{\sup_{t\leqq T}|f^{\varepsilon,N}(t)-f(x^{\varepsilon,N}(t))|\geqq\alpha\}\to 0 \quad \text{as } \varepsilon\to 0 \quad \text{for each } \alpha>0, \tag{3.9}$$

$$\sup_{t\leqq T}|\hat{A}^{\varepsilon,N}f^{\varepsilon,N}(t)|\leqq M_T^{\varepsilon,N}(f) \tag{3.10}$$

$$\sup_{\varepsilon>0} P\{M_T^{\varepsilon,N}(f)\geqq K\}\to 0 \quad \text{as} \quad K\to\infty. \tag{3.11}$$

Then $\{f(x^{\varepsilon,N}(\cdot))\}$ is tight in $D^1[0,\infty)$ for each $f(\cdot)\in\mathscr{D}_1$ and $\{x^{\varepsilon,N}(\cdot)\}$ is also tight in $D^r[0,\infty)$.

Remark. Normally, we let $\mathscr{D}_1$ be the subset of functions in $\mathscr{D}$ which do not depend on t, and the $f^{\varepsilon,N}(\cdot)$ would be constructed as they would be for use in Theorem 1. This is illustrated in Sect. 5, and also in [4, 5], where the sequence $\{f^\varepsilon(\cdot)\}$ of perturbations must be constructed in order to use Kurtz's original method. Equations (3.9)–(3.11) require more of the $f^{\varepsilon,N}(\cdot)$ than the p-lim requirements of Theorem 1. But, as implied above, these additional properties often exist. In the example of Sect. 5 $\hat{A}^{\varepsilon,N}f^{\varepsilon,N}(t)$ is bounded uniformly in t and the sup term in (3.9) goes to zero uniformly in ω (w.p.1) as $\varepsilon\to 0$.

Proof. In order to simplify the notation, *drop the superscript N*. By Lemma 1 and (3.9), for each $f(\cdot)\in\mathscr{D}_1$ and $T<\infty$, we need only find $\gamma_\varepsilon(\delta)$ satisfying (3.8) and such that for $u\leqq\delta$, $t\leqq T$,

$$E_t^\varepsilon\gamma_\varepsilon(\delta)\geqq\min[1, E_t^\varepsilon|f(x^\varepsilon(t+u))-f(x^\varepsilon(t))|^2].$$

---

[6] E_t^ε denotes conditioning on $y^\varepsilon(s)$, $s\leqq t$

Let $f(\cdot)\in\mathscr{D}_1$ and let $\{f^\varepsilon(\cdot)\}$ (resp., $\{f_2^\varepsilon(\cdot)\}$) denote the sequences in $\mathscr{D}(\hat{A}^\varepsilon)$ associated with $f(\cdot)$ (resp., with $f^2(\cdot)$) by the hypotheses. Write $f(x^\varepsilon(t))=f(t)$. Then

$$\begin{aligned}|f(t+u)-f(t)|^2=&(f_2^\varepsilon(t+u)-f_2^\varepsilon(t))-2f(t)(f^\varepsilon(t+u)-f^\varepsilon(t))\\&+(f^2(t+u)-f_2^\varepsilon(t+u))+(f_2^\varepsilon(t)-f^2(t))\\&+2f(t)[(f^\varepsilon(t+u)-f(t+u))-(f^\varepsilon(t)-f(t))].\end{aligned}$$

First suppose (Case 1): $\sup_\varepsilon EM_T^\varepsilon(f)+\sup_\varepsilon EM_T^\varepsilon(f^2)<\infty$ and $\sup_{t\le T}(|f^\varepsilon(t)-f(t)|+|f_2^\varepsilon(t)-f^2(t)|)\equiv\alpha^\varepsilon(T)\to 0$ uniformly in ω as $\varepsilon\to 0$. Then we can get a $\gamma_\varepsilon(\delta)$ of the desired type if there is a function $\bar{\gamma}_\varepsilon(\delta)$ satisfying (3.8) and such that for $u\le\delta$, $t\le T$,

$$E_t^\varepsilon\bar{\gamma}_\varepsilon(\delta)\ge 2\|f(\cdot)\|\,|E_t^\varepsilon f^\varepsilon(t+u)-f^\varepsilon(t)|+|E_t^\varepsilon f_2^\varepsilon(t+u)-f_2^\varepsilon(t)|.$$

By (3.3) and (3.10), we can use

$$\bar{\gamma}_\varepsilon(\delta)=\delta[2\|f(\cdot)\|M_T^\varepsilon(f)+M_T^\varepsilon(f^2)]$$

and the proof of Case 1 is completed.

Now, let the condition of Case 1 not hold and use a truncation argument. For each $\delta>0$ define

$$T_\delta^\varepsilon=\min\{t:|\hat{A}^\varepsilon f^\varepsilon(s)|\ge 1/\delta \text{ or } |\hat{A}^\varepsilon f_2^\varepsilon(s)|\ge 1/\delta \text{ or } \alpha^\varepsilon(t)\ge\delta\}.$$

Let τ_ε denote a stopping time (relative to $\{\mathscr{F}_t^\varepsilon\}$) and note [3] that (3.3) holds if t and $t+s$, resp., are replaced by $t\cap\tau_\varepsilon$ and $(t+s)\cap\tau_\varepsilon$, resp. Now repeat the Case 1 argument to get that $\{f(\cdot\cap T_\delta^\varepsilon)\}$ is tight for each $\delta>0$. Equations (3.9) and (3.11) (for both f and f^2) imply that $\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}P\{T_\delta^\varepsilon>T\}=1$. This and the tightness of $\{f(\cdot\cap T_\delta^\varepsilon)\}$ for each $\delta>0$ completes the proof.

4. Discrete Parameter Processes

Only a sketch will be given. For each $\varepsilon>0$, let $\{x_i^\varepsilon\}$ denote a discrete parameter R^r valued process and let $\{\tau_\varepsilon\}$ denote a sequence of positive numbers tending to zero. Define the interpolation $x^\varepsilon(\cdot)$ by $x^\varepsilon(t)=x_i^\varepsilon$ on $[i\tau_\varepsilon, i\tau_\varepsilon+\tau_\varepsilon)$. Let $x^{\varepsilon,N}(\cdot)$ be defined the same way it was defined in Theorem 1. For $f(\cdot)\in\mathscr{D}$ define $\hat{A}^{\varepsilon,N}f(\cdot)$ by

$$\hat{A}^{\varepsilon,N}f(t)=[E_t^{\varepsilon,N}f(t+\tau_\varepsilon)-f(t)]/\tau_\varepsilon. \tag{4.1}$$

Then

$$E_{i\tau_\varepsilon}^{\varepsilon,N}f((i+n)\tau_\varepsilon)-f(i\tau_\varepsilon)=\sum_{j=i}^{i+n-1}E_{i\tau_\varepsilon}^{\varepsilon,N}\hat{A}^{\varepsilon,N}f(j\tau_\varepsilon)\tau_\varepsilon. \tag{4.2}$$

The suggestion for adapting the continuous parameter results to the discrete parameter case which was made in [8, p. 625] can be hard to use, because with that approach the construction of the $\{f^\varepsilon(\cdot)\}$ or $\{f^{\varepsilon,N}(\cdot)\}$ can be quite messy. It often seems best to deal with the discrete parameter case directly, using (4.2) and

the martingale method of Theorem 1. In many cases, our method seems to simplify the problem of obtaining appropriate $\{f^\varepsilon(\cdot)\}$ or $\{f^{\varepsilon,N}(\cdot)\}$. For example, for the discrete parameter analog of the problems in [4, 1] or in Sect. 5 below, $f^\varepsilon(\cdot)$ (or $f^{\varepsilon,N}(\cdot)$) takes the form

$$f^\varepsilon(\cdot)=f(\cdot)+f_1^\varepsilon(\cdot)+f_2^\varepsilon(\cdot),$$

where all functions are constant on intervals $[i\tau_\varepsilon, i\tau_\varepsilon+\tau_\varepsilon)$ and where the $f_i(\cdot)$ are constructed analogously to the method used in [4], [1], or as in Sect. 5 below, but where the appropriate series are used in lieu of integrals. A simpler treatment of the discrete parameter case was one of the motivations for the approach of this paper. Examples will appear in a subsequent paper. Using the methods of [8, 4], reference [5] shows how to get the $f^\varepsilon(\cdot)$ from $f(\cdot)$ for one type of discrete parameter case.

Theorem 3. *Assume the conditions of Theorem* 1, *except that the* $f^{\varepsilon,N}(\cdot)$ *are constant on the intervals* $[i\tau_\varepsilon, i\tau_\varepsilon+\tau_\varepsilon)$, *and the "discrete parameter"* $\hat{A}^{\varepsilon,N}$ *of* (4.1) *is used. Then the conclusions of Theorem* 1 *hold.*

The proof is almost identical to that of Theorem 1 and is omitted. We need only work with (3.3) with t, s being integral multiples of τ_ε. Then (3.3) reduces to (4.2).

Theorem 2 can be used to prove tightness and the "sups" are over points $i\tau_\varepsilon \leqq T$, $i=0,1,\ldots$, only.

5. An Application to Averaging

In order to illustrate the ease of use of Theorem 1, we redo the classical averaging problem of Hazminskij [2]. The conditions are not the best and can easily be weakened. Let $\xi(\cdot)$ be a R^q valued stationary bounded right continuous ϕ-mixing process with rate satisfying $\int_0^\infty \phi^{1/2}(s)\,ds<\infty$. Let $F(\cdot,\cdot,\cdot)$ be a continuous R^r valued function on R^{r+1+q}, whose first partial x-derivatives are continuous. Define $\bar{F}(x,t)=EF(x,t,\xi)$. Let $q_N(\cdot)$ be as below (3.6a). Define $\xi^\varepsilon(t)=\xi(t/\varepsilon)$, $F_N(x,t,\xi)=F(x,t,\xi)q_N(\xi)$, $\bar{F}_N(x,t)=\bar{F}(x,t)q_N(x)$ and define $x^\varepsilon(\cdot)$ and $x^{\varepsilon,N}(\cdot)$ by

$$\dot{x}^\varepsilon=F(x^\varepsilon,t,\xi^\varepsilon) \tag{5.1a}$$

$$\dot{x}^{\varepsilon,N}=F(x^{\varepsilon,N},t,\xi^\varepsilon)q_N(x^{\varepsilon,N})=F_N(x^{\varepsilon,N},t,\xi^\varepsilon). \tag{5.1b}$$

In Theorems 4 and 5, $E_t^{\varepsilon,N}$ denotes expectation conditioned on $\mathscr{F}_t^\varepsilon$, the minimal σ-algebra measuring $\{\xi^\varepsilon(s), s\leqq t, x^\varepsilon(0)\}$.

Theorem 4. *Assume the conditions of the previous paragraph, let* $x^\varepsilon(0)\to x_0$ *weakly and suppose that* $\dot{x}=\bar{F}(x,t)$ *has a unique (bounded on bounded intervals) solution* $\bar{x}(\cdot)$ *for each* $x(0)\in R^r$. *Then* $x^\varepsilon(\cdot)\to\bar{x}(\cdot)$ *weakly, where* $\bar{x}(0)=x_0$.

Proof. The proof is a very straightforward application of Theorem 1. Fix N. The $\{x^{\varepsilon,N}(\cdot)\}$ is tight since $|\dot{x}^{\varepsilon,N}(t)|$ is bounded uniformly in ε, t. Fix $f(\cdot,\cdot)\in\mathscr{C}_0^{1,2}$. Then

$\hat{A}^{\varepsilon,N} f(x^{\varepsilon,N}(t),t) = f_t(x^{\varepsilon,N}(t),t) + f'_x(x^{\varepsilon,N}(t),t) F_N(x^{\varepsilon,N}(t),t,\xi^\varepsilon(t))$. Define $f_1^{\varepsilon,N}(t) = f_1^{\varepsilon,N}(x^{\varepsilon,N}(t),t)$, where

$$f_1^{\varepsilon,N}(x,t) = \int_0^\infty E_t^{\varepsilon,N} f'_x(x,t+s)[F_N(x,t+s,\xi^\varepsilon(t+s)) - \bar{F}_N(x,t+s)]\,ds$$
$$= \varepsilon \int_0^\infty E_t^{\varepsilon,N} f'_x(x,t+\varepsilon s)\left[F_N\left(x,t+\varepsilon s,\xi\left(\frac{t}{\varepsilon}+s\right)\right) - \bar{F}_N(x,t+\varepsilon s)\right] ds.$$

By the compact support of $f(\cdot,\cdot)$, the centering of the integrand about its mean and the ϕ-mixing condition,

$$f_1^{\varepsilon,N}(t) = O(\varepsilon) \quad \textit{uniformly in } t,\omega. \tag{5.2}$$

The gradient $f_{1,x}^{\varepsilon,N}(x,t)$ also exists, and the differentiation can be done under the integral sign. Also, $f_{1,x}^{\varepsilon,N}(x^{\varepsilon,N}(t),t) = O(\varepsilon)$ uniformly in t,ω.

Define the operator A^N by $A^N f(x,t) = \bar{F}'_N(x,t) f_x(x,t)$. Define the function $f^{\varepsilon,N}(\cdot)$ by $f^{\varepsilon,N}(t) = f(x^{\varepsilon,N}(t),t) + f_1^{\varepsilon,N}(t)$. Then $f^{\varepsilon,N}(\cdot) \in \mathscr{D}(\hat{A}^{\varepsilon,N})$ and (abbreviate $x = x^{\varepsilon,N}(t)$)

$$\begin{aligned}\hat{A}^{\varepsilon,N} f^{\varepsilon,N}(t) &= f_t(x,t) + f'_x(x,t) F_N(x,t,\xi^\varepsilon(t)) \\ &\quad - f'_x(x,t)[F_N(x,t,\xi^\varepsilon(t)) - \bar{F}_N(x,t)] + (f_{1,x}^{\varepsilon,N}(x,t)) F_N(x,t,\xi^\varepsilon(t)) \\ &= \left(\frac{\partial}{\partial t} + A^N\right) f(x,t) + O(\varepsilon).\end{aligned} \tag{5.3}$$

Equations (5.2) and (5.3) and the uniqueness of $\bar{x}(\cdot)$ imply the conditions of Theorem 1. Q.E.D.

Linearization and first order perturbations. We now examine the asymptotic properties of the centered solution $x^\varepsilon(t) - \bar{x}(\cdot) = \delta x^\varepsilon(t)$, as $\varepsilon \to 0$.

Theorem 5. *Assume that $F(\cdot,t,\xi)$ has two continuous in (x,t,ξ) partial x-derivatives, and that $\dfrac{x^\varepsilon(0) - x_0}{\sqrt{\varepsilon}} \to 0$ weakly. Then under the conditions of Theorem 4, $u^\varepsilon(\cdot) = (x^\varepsilon(\cdot) - \bar{x}(\cdot))/\sqrt{\varepsilon}$ converges weakly to the solution of the stochastic differential equation* (5.4)

$$du = \bar{F}_x(\bar{x}(t),t)u\,dt + dB, \qquad u(0) = 0, \tag{5.4a}$$

where $B(\cdot)$ is a Brownian motion with covariance

$$\frac{d \cdot EB(t)B'(t)}{dt} = \int_{-\infty}^{\infty} E(F(\bar{x}(t),t,\xi(0)) - \bar{F}(\bar{x}(t),t))(F(\bar{x}(t),t,\xi(s)) - \bar{F}(\bar{x}(t),t))'\,ds. \tag{5.4b}$$

Proof. We have

$$\dot{u}^\varepsilon(t) = \frac{F(x^\varepsilon(t),t,\xi^\varepsilon(t)) - F(\bar{x}(t),t,\xi^\varepsilon(t))}{\sqrt{\varepsilon}} + \frac{\delta F^\varepsilon(t)}{\sqrt{\varepsilon}} \tag{5.5}$$

where $\delta F^\varepsilon(t) = F(\bar{x}(t), t, \xi^\varepsilon(t) - \bar{F}(\bar{x}(t), t)$. Although $x(0) = \bar{x}_0$ might be random it plays no important role in the proof and we suppose for convenience that it is constant and N will be large enough such that $|x_0| \leqq N$. We need only work on an arbitrary finite interval $[0, T]$ and hence can suppose that $|\bar{x}(t)|$ is bounded in t. Also, by a suitable choice of probability space, we can suppose that $\bar{x}^\varepsilon(0) \to x_0$ w.p.l. Thus, since $\bar{x}(\cdot)$ is determinisitic and $x^\varepsilon(\cdot) \to \bar{x}(\cdot)$ weakly, for each $\delta > 0$ there is an $\varepsilon_\delta > 0$ such that $P\{\sup_{t \leqq T} |x^\varepsilon(t) - \bar{x}(t)| \geqq \delta\} \leqq \delta$ for $\varepsilon \leqq \varepsilon_\delta$. In order to prove (5.4) we can w.l.o.g., suppose that for an arbitrarily small but fixed $\delta > 0$, $\sup_{t \leqq T} |x^\varepsilon(t) - \bar{x}(t)| \leqq \delta$.

Using these simplifications and expanding (5.5) yields

$$\dot{u}^\varepsilon = F_x(\bar{x}, t, \xi^\varepsilon) u^\varepsilon + \sqrt{\varepsilon}\,(u^\varepsilon)' F^\varepsilon_{xx}(t) u^\varepsilon + \delta F^\varepsilon(t)/\sqrt{\varepsilon},$$

where $F^\varepsilon_{xx}(t)$ is a matrix which is bounded on $[0, T]$. Define $u^{\varepsilon,N}(\cdot)$ by $u^{\varepsilon,N}(0) = u^\varepsilon(0)$ when $|u^\varepsilon(0)| \leqq N$ and zero otherwise, and for $t > 0$ by

$$\begin{aligned} \dot{u}^{\varepsilon,N} = {} & [F_x(\bar{x}, t, \xi^\varepsilon) u^{\varepsilon,N} + \sqrt{\varepsilon}\,(u^{\varepsilon,N})' F^\varepsilon_{xx}(t) u^{\varepsilon,N}] q_N(u^{\varepsilon,N}) \\ & + \delta F^\varepsilon(t)/\sqrt{\varepsilon}. \end{aligned} \tag{5.6}$$

If $B^\varepsilon(t) = \int_0^t \delta F^\varepsilon(s)\,ds/\sqrt{\varepsilon}$ is tight on $D^r[0, \infty)$ with all weak limits being the same *Wiener* processes, than we note merely in passing, that (5.6) implies that $\{u^{\varepsilon,N}(\cdot)\}$ is tight for each N and that all weak limits have continuous paths.

Proceeding, let $f(\cdot, \cdot) \in \mathscr{C}_0^{1,3}$. Then

$$\hat{A}^{\varepsilon,N} f(u^{\varepsilon,N}(t), t) = f_t(u^{\varepsilon,N}(t), t) + f'_u(u^{\varepsilon,N}(t), t) \cdot \dot{u}^{\varepsilon,N}(t).$$

The component

$$\sqrt{\varepsilon}\, f'_u(u^{\varepsilon,N}(t), t) q_N(u^{\varepsilon,N}(t)) [(u^{\varepsilon,N}(t))' F^\varepsilon_{xx}(t) u^{\varepsilon,N}(t)]$$

of the second term above always has $p\lim_{\varepsilon \to 0} = 0$. We must find a sequence $\{f^{\varepsilon,N}(\cdot)\}$ satisfying the conditions of Theorem 1. We look for $f^{\varepsilon,N}(\cdot)$ of the form

$$f^{\varepsilon,N}(t) = f(u^{\varepsilon,N}(t), t) + \sum_{i=1}^{2} f_i^{\varepsilon,N}(t)$$

where the $f_i^{\varepsilon,N}(t)$ have the form $f_i^{\varepsilon,N}(u^{\varepsilon,N}(t), t)$ and the $f_i^{\varepsilon,N}(u, t)$ are now to be defined.

$$\begin{aligned} f_1^{\varepsilon,N}(u, t) = {} & \int_0^\infty E_t^{\varepsilon,N} f'_u(u, t+s) \left[(F_x(\bar{x}(t+s), t+s, \xi^\varepsilon(t+s)) \right. \\ & \left. - \bar{F}_x(\bar{x}(t+s), t+s)) q_N(u) + \frac{\delta F^\varepsilon(t+s)}{\sqrt{\varepsilon}} \right] ds \\ = {} & \varepsilon \int_0^\infty E_t^{\varepsilon,N} f'_u(u, t+\varepsilon s) \left[F_x\left(\bar{x}(t+\varepsilon s), t+\varepsilon s, \xi\left(\frac{t}{\varepsilon} + s\right)\right)\right. \\ & \left. - \bar{F}_x(\bar{x}(t+\varepsilon s), t+\varepsilon s)) q_N(u) + \delta F^\varepsilon(t+\varepsilon s)/\sqrt{\varepsilon} \right] ds. \end{aligned} \tag{5.7}$$

By the compact support of $f(\cdot,\cdot)$, the centering of the integrand about its expectation and the ϕ-mixing, $p\text{-}\lim_{\varepsilon\to 0} f_1^{\varepsilon,N}(t)=0$. Also, $f_1^{\varepsilon,N}(\cdot,t)$ is differentiable, and we can take the derivative under the integral sign. Using this, we readily show that $f_1^{\varepsilon,N}(\cdot)\in\mathscr{D}(\hat{A}^{\varepsilon,N})$ and (abbreviate $u=u^{\varepsilon,N}(t)$)

$$\begin{aligned}\hat{A}^{\varepsilon,N}f_1^{\varepsilon,N}(t)=&-f_u'(u,t)[(F_x(\bar{x}(t),t,\xi^\varepsilon(t))-\bar{F}_x(\bar{x}(t),t))q_N(u)\\&+\delta F^\varepsilon(t)/\sqrt{\varepsilon}]+f_{1,u}^{\varepsilon,N}(u,t)'\dot{u}^{\varepsilon,N}(t).\end{aligned}\tag{5.8}$$

The only component of the last term of (5.8) which does not have $p\text{-}\lim=0$ is that arising from the $\delta F^\varepsilon/\sqrt{\varepsilon}$ part of $\dot{u}^{\varepsilon,N}$ and is ($u^{\varepsilon,N}$ still written as u) $\overset{}{}_{\varepsilon\to 0}$

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^\infty E_t^{\varepsilon,N}\delta F^{\varepsilon'}(t+s)f_{uu}(u,t+s)\delta F^\varepsilon(t)ds\\&\quad=\int_0^\infty E_t^{\varepsilon,N}\delta F^{\varepsilon'}(t+\varepsilon s)f_{uu}(u,t+\varepsilon s)\delta F^\varepsilon(t)ds\equiv Q^{\varepsilon,N}f(u,t)\end{aligned}$$

which is bounded uniformly in ω,t by the ϕ-mixing. Define the operator A_0^ε by $EQ^{\varepsilon,N}f(u,t)=A_0^\varepsilon f(u,t)$. By the stationarity of $\xi(\cdot)$ and the ϕ-mixing we have the following convergence uniformly in u,t, where $\delta F(t,s)=F(\bar{x}(t),t,\xi(t+s))-\bar{F}(\bar{x}(t),t)$ and the integral defines the operator A_0:

$$A_0^\varepsilon f(u,t)\to\int_0^\infty E\delta F'(t,s))f_{uu}(u,t)\delta F(t,0)ds=A_0f(u,t)$$

Next, in order to "average out" the $Q^{\varepsilon,N}$ term, we introduce $f_2^{\varepsilon,N}(u,t)$ in the form

$$\begin{aligned}f_2^{\varepsilon,N}(u,t)=\int_0^\infty ds E_t^{\varepsilon,N}\Big\{&\frac{1}{\varepsilon}\int_0^\infty dv E_{t+s}^{\varepsilon,N}\delta F^{\varepsilon'}(t+s+v)f_{uu}(u,t+s+v)\delta F^\varepsilon(t+s)\\&-A_0^\varepsilon f(u,t+s)\Big\}.\end{aligned}$$

By the change of variables $s/\varepsilon\to s$, $v/\varepsilon\to v$, and the ϕ-mixing condition, it is readily seen that $f_2^{\varepsilon,N}(u,t)=O(\varepsilon)$ uniformly in ω,t,u. Also, $f_2^{\varepsilon,N}(t)=f_2^{\varepsilon,N}(u^{\varepsilon,N}(t),t)\in\mathscr{D}(\hat{A}^{\varepsilon,N})$ and

$$\hat{A}^{\varepsilon,N}f_2^{\varepsilon,N}(t)=-Q^{\varepsilon,N}f(u^{\varepsilon,N}(t),t)+A_0^\varepsilon f(u^{\varepsilon,N}(t),t)+O(\sqrt{\varepsilon}),$$

where $O(\sqrt{\varepsilon})$ is uniform in ω,t.

Adding all the above yields

$$p\text{-}\lim_{\varepsilon\to 0}[f^{\varepsilon,N}(\cdot)-f(u^{\varepsilon,N}(\cdot),\cdot)]=0\tag{5.9}$$

$$\begin{aligned}p\text{-}\lim_{\varepsilon\to 0}[\hat{A}^{\varepsilon,N}f^{\varepsilon,N}(\cdot)-\{&f_t(u^{\varepsilon,N}(\cdot),\cdot)\\&+f_u'(u^{\varepsilon,N}(\cdot),\cdot)q_N(u^{\varepsilon,N}(\cdot))\bar{F}_x(\bar{x}(\cdot),\cdot)u^{\varepsilon,N}(\cdot)\\&+A_0f(u^{\varepsilon,N}(\cdot),\cdot)\}]=0.\end{aligned}\tag{5.10}$$

If $\{u^{\varepsilon,N}(\cdot)\}$ or $\{B^{\varepsilon}(\cdot)\}$ were tight, then via Theorem 1 any weak limit of $\{u^{\varepsilon,N}(\cdot)\}$ would be a diffusion with operator A^N where

$$A^N f(u,t) = f'_u(u,t) q_N(u) \bar{F}_x(\bar{x}(t),t) u + A_0 f(u,t) \tag{5.11}$$

and Theorem 1 would give us the limit (5.4). Then, to get the covariance form given in (5.4b) from (5.11), we just symmetrize the coefficients in A_0: Write $A_0 f(u,t) = \frac{1}{2}\sum_{i,j} a_{ij}(t) f_{u_i u_j}(u,t)$, where we take the symmetric form for $\{a_{ij}(\cdot)\} \equiv \Sigma(\cdot)$ (i.e., $a_{ij}(\cdot) = a_{ji}(\cdot)$). Then note that

$$(E\delta F(t,s)\delta F'(t,0))' = E\delta F(t,-s)\delta F'(t,0),$$

and use $A_0 f(u,t) = \frac{1}{2}$ trace $\Sigma(t) f_{uu}(u,t)$ and $\frac{d}{dt} EB(t)B'(t) = \Sigma(t)$.

The tightness of $\{u^{\varepsilon,N}(\cdot)\}$ could be proved either by proving it directly for $\{u^{\varepsilon,N}(\cdot)\}$, or for the $\{B^{\varepsilon}(\cdot)\}$ and then using the remark below (5.6). To prove the weak convergence of $\{B^{\varepsilon}(\cdot)\}$ to the desired Wiener process, we could work with $B^{\varepsilon,N}(\cdot)$ defined by $\dot{B}^{\varepsilon,N} = \delta F^{\varepsilon}(t) q_N(B^{\varepsilon,N})/\sqrt{\varepsilon}$, $B^{\varepsilon,N}(0) = 0$, and proceed exactly as for the $\{u^{\varepsilon,N}(\cdot)\}$ above to show the desired convergence. An alternative method for both tightness and weak convergence follow the lines of the preceding development but with the δF^{ε} in (5.6) multiplied by $q_N(u^{\varepsilon,N})$ also. Then $\{u^{\varepsilon,N}(\cdot)\}$ is bounded for each N and tightness can be proved by the method of Theorem 2. The details are the same as for the given development, except that a term proportional to $q_{N,u}(u)$ appears in the expression for the weak limit of $\{u^{\varepsilon,N}(\cdot), \varepsilon > 0\}$. But this term is zero for $|u| < N$ and disappears when $N \to \infty$. For this method the f_i^{ε} satisfy the same bounds ($O(\sqrt{\varepsilon})$) as they do in the given development and the $M_T^{\varepsilon,N}(f)$ are bounded. Thus, Theorem 2 implies tightness. We omit the details. Q.E.D.

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Sur les instants de grande amplitude des trajectoires de processus gaussiens stationnaires

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§0. Introduction

Considérons un processus gaussien stationnaire $X(t)$, $t \geqq 0$, centré, séparable, de covariance $r(t)$ continue avec $r(0)=1$.

On suppose de plus, dans toute la suite,

(H_1) le processus X est presque sûrement majoré sur toute partie bornée de $\mathbb{R}_+$,

(H_2) la covariance $r(t)$ tend vers zéro quand t tend vers l'infini.

Soit en outre une application, croissante, $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ tendant vers l'infini avec la variable. Nous avons montré dans [3] que si (H_2) est renforcée en

$$(H_2') \qquad r(u) = O(1/\log u), \quad u \to \infty,$$

alors pour toute φ telle que

a) l'intégrale $\int^{+\infty} e^{-\frac{1}{2}\varphi^2(u)} \dfrac{du}{\varphi(u)}$ diverge,

b) $\varphi(t) \overset{\cup}{\cap}{}^{1} \sqrt{\log t}$ quand t tend vers l'infini; la mesure de Lebesgue de l'ensemble aléatoire $A_\varphi(\omega) = \{t \geqq 0 : X(\omega, t) > \varphi(t)\}$ est presque sûrement infinie. Ce résultat est une conséquence immédiate d'une propriété plus forte que nous avions établie dans [3], (th. 7.1 p. 58). Nous avions montré, en conservant les hypothèses ci-dessus,

$$(1) \qquad \forall S > 0, \quad P\left\{\omega : \overline{\lim_{T \to \infty}} \frac{\lambda\{A_\varphi(\omega) \cap [S, T]\}}{\int_S^T \psi_0 \varphi(u)\, du} \geqq 1\right\} = 1,$$

où λ est la mesure de Lebesgue et $\psi(x) = P\{\mathcal{N}(0,1) > x\}$.

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[1] $f(t) \overset{\cup}{\cap} g(t)$ en 0 (resp $+\infty$) si $0 < \varliminf_{t \to 0 (+\infty)} \left|\frac{f(t)}{g(t)}\right| \leqq \varlimsup_{t \to 0 (+\infty)} \left|\frac{f(t)}{g(t)}\right| < \infty$

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(2) Si φ est croissante et vérifie seulement b), alors

$$P\left\{\omega: \overline{\lim_{T\to\infty}} \frac{\lambda\{A_\varphi^c(\omega)\cap[0,T]\}}{T}=1\right\}=1.$$

Ces résultats s'obtiennent simplement. On peut les récrire presque identiquement en substituant à $A_\varphi(\omega)$, l'ensemble $\bar{A}_\varphi(\omega)=\{t\geqq 0: |X(\omega,t)|>\varphi(t)\}$. Les nouveaux énoncés différeront des précédents d'un facteur $\sqrt{2/\pi}$ provenant de la loi de $|X(t)|$.

Nous proposons dans cet article de renforcer ces propriétés en examinant non plus la répartition sur l'axe des temps des grandes amplitudes de la trajectoire $X(\omega)$ mais plutôt la durée de celles-ci. Nous commencons par poser

$$\mu(X,\varphi)=\mathrm{Sup}\{\alpha\geqq 0: \forall T, P\{\exists t>T: \forall u\in]t, t+\alpha[, |X(u)|>\varphi(u)\}=1\}.$$

Nous montrons sous des conditions assez générales que

$$\mu(X,\varphi)>0.$$

Ces conditions sont, d'une part, les hypothèses (H_1) et (H_2) sur X, et d'autre part, $\varphi(t)=(1-\varepsilon)\sqrt{2\log t}$, $0<\varepsilon\leqq 1$.

Nous montrons aussi qu'il existe une classe de processus gaussiens telle que $\mu(X,\varphi)$ est arbitrairement grand. Cette classe est constituée des processus gaussiens stationnaires tels que $r(t)$ soit deux fois dérivable en $t=0$. Posons alors

$$\mu(X,\varepsilon)=\mu(X,(1-\varepsilon)\sqrt{2\log t}).$$

Nous établissons dans le théorème 2

$$\mu(X,\varepsilon)\geqq\frac{\varepsilon}{C_0\sqrt{-r''(0)\log\dfrac{C_0}{\varepsilon}}},$$

où $C_0>0$ est une constante absolue.

Ces résultats reposent avant tout sur une estimation de la loi de $\inf_{0\leqq t\leqq 1}|X(t)|$, estimation qui fait l'objet du paragraphe suivant.

§1. Lemmes préliminaires

Considérons un processus gaussien centré $\{X(t), t\in T\}$ où T est un ensemble arbitraire. Notons d l'écart sur T associé à X. Nous posons pour tout $u>0$, $M_d(T,u)$ le cardinal maximal des suites $S\subset T$ telles que

(*) $\qquad \forall s,t\in S,\quad \text{si}\quad s\neq t\quad \text{alors}\quad d(s,t)>u.$

Lorsque $M_d(T,u)$ est fini, nous notons $S_d(T,u)$ une des suites maximales pour (*). Nous faisons les hypothèses suivantes

$h-0$, le processus X est d-séparable,
$h-1$, $\forall t\in T, E\{X^2(t)\}=1$,
$h-2$, $\delta=\sup_{T\times T} d(s,t)<\sqrt{2}$,
$h-3$, l'intégrale $\int_{+0}\sqrt{\log M_d(T,u)}\,du$ converge,
$h-4$, il existe un réel ρ compris entre 0 et 1 strictement tel que

$$Y(\rho,\delta)=\sum_{k=1}^{\infty}\left[(\rho^{k-1}\delta)^2\vee\left(2\delta\rho^{k-1}\sqrt{\log\frac{M_d(T,\rho^k\delta)}{\rho^k\delta}}\right)\right]<1;$$

Mettons en place les éléments suivants: pour tout $k\geqq 0$ et ρ fixé par $h-4$,

- $u_k=\rho^k\delta$,
- $\beta_k=S_d(T,u_k)$,
- $S=\bigcup_{k=0}^{\infty}\beta_k$ et pour tout s élément de S,
- $\eta(s)=\inf\{k\geqq 0: s\in\beta_k\}$.
- $y_k=(\rho^{k-1}\delta)^2\vee\left(2\delta\rho^{k-1}\sqrt{\log\frac{M_d(T,u_k)}{u_k}}\right),\quad k\geqq 1.$

Remarquons que β_0 est nécessairement réduit à un point que nous noterons dès à présent σ. Nous construisons enfin une application bornée $m: T\to\mathbb{R}_+$ définie par

$$m(s)=\begin{cases}0 & \text{si } s\in S \text{ et } \eta(s)=0,\\ y_1+\ldots+y_{\eta(s)} & \text{si } s\in S \text{ et } \eta(s)>0,\\ Y(\rho,\delta) & \text{si } s\in T\setminus S,\end{cases}$$

et nous posons pour tout s élément de T, $l(s)=\dfrac{1}{1-m(s)}$.

Lemme 1. *Sous les hypothèses* (h: 0−4), *pour tout* $\lambda>\sqrt{2}$, *nous avons*

$$P\{\inf_{s\in T}|l(s)\cdot X(s)|<\lambda \quad\text{et}\quad |X(\sigma)|>\lambda\}$$
$$\leqq 0(1)\,\Psi(\lambda(1-Y(\rho,\delta)))\,\frac{(\delta\rho)^{\frac{\lambda^2}{2}}}{1-\rho^{\frac{\lambda^2}{2}}}.$$

Démonstration. Posons pour tout $k\geqq 0$, $A_k=\{\inf_{\beta_k}|l\cdot X|<\lambda\}$.

Par suite de la d-séparabilité de X, il nous suffit de majorer

$$P\left\{\bigcup_{k=1}^{\infty}A_k\cap A_0^c\right\}\leqq\sum_{\substack{k=1\\ s\in\bar\beta_k}}^{\infty}\alpha_k(s),$$

où $\bar\beta_k=\beta_k\cap\beta_{k-1}^c\cap\ldots\cap\beta_0^c$, et

$$\alpha_k(s)=P\{|l(s)\,X(s)|<\lambda,\ |X(\tau_k(s))\,l(\tau_k(s))|\geqq\lambda\},$$

où $\tau_k: \beta_k \to \beta_{k-1}$ est une des applications définies par $d(s, \tau_k(s)) \leqq u_{k-1}$. Quand s parcourt $\bar{\beta}_k$, $\eta(s)$ est égal à k; ainsi

$$m(s) = \sum_{b=1}^{k} y_b,$$

et, plus simplement,

$$m(\tau_k(s)) \leqq \sum_{b=1}^{k-1} y_b \quad \text{si} \quad k>1, \quad (=0 \text{ si } k=1).$$

De sorte que $\alpha_k(s)$ se majore par

$$P\left\{|X(s)| < \left(1-\sum_{b=1}^{k} y_b\right)\lambda,\ |X(\tau_k(s))| \geqq \left(1-\sum_{b=1}^{k-1} y_b\right)\lambda\right\}.$$

Décomposons $X(s)$ suivant $X(\tau_k(s))$,

$$X(s) = \eta X(\tau_k(s)) + \sqrt{1-\eta^2}\, Z$$

où Z est une v.a. indépendante de $X(\tau_k(s))$, Z suit une loi $\mathcal{N}(0, 1)$ et où le facteur η est déterminé par

$$\eta = E\{X(s) \cdot X(\tau_k(s))\} = 1 - \tfrac{1}{2} d^2(s, \tau_k(s)) \geqq 1 - \tfrac{1}{2}\delta^2 > 0,$$

par $h-2$.

A partir de la relation

$$|Z\sqrt{1-\eta^2}| \geqq |\eta X(\tau_k(s))| - |X(s)|,$$

nous déduisons la majoration

$$\alpha_k(s) \leqq P\left\{|X(\tau_k(s))| \geqq \lambda\left(1-\sum_{b=1}^{k-1} y_b\right),\ |Z\sqrt{1-\eta^2}| > \lambda(y_k - \tfrac{1}{2}(\delta\rho^{k-1})^2)\right\}$$

dans laquelle $y_k - \frac{1}{2}(\rho^{k-1}\delta)^2 > 0$, pour tout $k \geqq 1$. D'où

$$\alpha_k(s) \leqq \Psi(\lambda(1-Y))\ \Psi\left(\lambda\sqrt{\log \frac{M_d(T, u_k)}{u_k}}\right),$$

$$\leqq 0(1)\ \Psi(\lambda(1-Y))[M_d(T, u_k)]^{-\frac{\lambda^2}{2}} (u_k)^{\frac{\lambda^2}{2}}.$$

Il en résulte la majoration suivante,

$$R_0 = \sum_{\substack{k=1 \\ s\in\bar{\beta}_k}}^{\infty} \alpha_k(s) \leqq 0(1)\ \Psi(\lambda(1-Y)) \sum_{k=1}^{\infty} [M_d(T, u_k)]^{1-\frac{\lambda^2}{2}} [\rho^k \delta]^{\frac{\lambda^2}{2}},$$

d'où, puisque $\lambda > \sqrt{2}$,

$$R_0 \leqq O(1)\ \Psi(\lambda(1-Y))\ \frac{(\delta\rho)^{\frac{\lambda^2}{2}}}{1-\rho^{\frac{\lambda^2}{2}}},$$

ce qui achève la démonstration du lemme 1.

Lemme 2. *Soit $\{X(t), t\in T\}$ un processus gaussien centré d'écart d. Nous supposons aussi que les hypothèses $(h-0)$, $(h-1)$ et $(h-3)$ du lemme 1 sont satisfaites. Alors pour ε compris entre 0 et 1 strictement on peut trouver deux nombres $0<B(\varepsilon)<\infty$ et $0<z(\varepsilon)<\infty$, tels que*

quels que soient $\lambda>\sqrt{2}$, $0<z\leqq z(\varepsilon)$ et $\sigma\in T$, on ait,

$$P\{\inf\{|X(s)|,\ s\in B_d(\sigma, z)\}<\lambda(1-\varepsilon),\ |X(\sigma)|>\lambda\}\leqq B(\varepsilon)\,\Psi(\lambda(1+\varepsilon)).$$

Démonstration. Pour tous ε, ρ compris entre 0 et 1 strictement, tout σ élément de T et $z>0$, nous posons

$$-Y(\rho,\sigma,z)=\sum_{k=1}^{\infty}(\rho^{k-1}z)^2\vee\left(2z\,\rho^{k-1}\sqrt{\log\frac{M_d(B_d(\sigma,z),\rho^k z)}{\rho^k z}}\right),$$

$$-H(\rho,\sigma,z)=\frac{2}{\rho^2(1-\rho^2)}\int_0^{\rho z}\left[u\vee\sqrt{\log\frac{M_d(B(\sigma,z),u)}{u}}\right]du,$$

$$-H(\rho,T,z)=\frac{2}{\rho^2(1-\rho^2)}\int_0^{\rho z}\left[u\vee\sqrt{\log\frac{M_d(T,u)}{u}}\right]du.$$

Un calcul simple permet de montrer

$$Y(\rho,\sigma,z)\leqq H(\rho,\sigma,z)\leqq H(\rho,T,z).$$

De plus, ces nombres sont tous finis par $(h-3)$. Nous constatons aussi que $H(\rho,T,.)$ est croissante et $H(\rho,T,0)=0$. Nous fixons à présent ρ dans l'intervalle $]0,\exp(-4\varepsilon)[$, par exemple $\rho=\bar{e}^4$, et nous posons,

$$\forall 0<\varepsilon<1,\qquad z(\varepsilon)=\inf\{z>0:\ H(\rho,T,z)>\varepsilon\}.$$

Alors pour tout $0<z\leqq z(\varepsilon)$ et tout σ élément de T, $Y(\rho,\sigma,z)$ est majoré par ε; et par conséquent

$$\begin{aligned}E&=P\{\inf\{|X(s)|,\ s\in B(\sigma,z)\}<\lambda(1-\varepsilon),\ |X(\sigma)|>\lambda\}\\&\leqq P\{\inf\{|X(s)|,\ s\in B(\sigma,z)\}<\lambda(1-Y(\rho,\sigma,z)),\ |X(\sigma)|>\lambda\},\\&\leqq P\{\inf\{|l_{\sigma,z}(s)\,X(s)|,\ s\in B(\sigma,z)\}<\lambda,\ |X(\sigma)|>\lambda\},\end{aligned}$$

où $l_{\sigma,z}(.)$ est la fonction $l(.)$ du lemme 1 correspondant à la boule $B(\sigma,z)$. Le lemme 1, appliqué à notre situation, permet d'écrire

$$\begin{aligned}E&\leqq O(1)\,\Psi(\lambda(1-Y))\frac{(z\rho)^{\frac{\lambda^2}{2}}}{1-\rho^{\frac{\lambda^2}{2}}},\\&\leqq O(1)\,\Psi[\lambda(1-\varepsilon)]\,[z(\varepsilon)\,\rho]^{\frac{\lambda^2}{2}}\,[1-\rho^{\frac{\lambda^2}{2}}]^{-1},\\&\leqq O(1)\,\Psi[\lambda(1+\varepsilon)]\,[1+2\varepsilon]\,[1-\rho^{\frac{\lambda^2}{2}}]^{-1},\\&\leqq O(1)\,\Psi[\lambda(1+\varepsilon)]\,(1+2\varepsilon),\end{aligned}$$

car $z(\varepsilon)\leqq\sqrt{\varepsilon(1-\rho^2)}<1$.

D'où le résultat annoncé.

§ 2. Application

Considérons un processus gaussien stationnaire $X(t)$, $t \geqq 0$, centré, séparable, de covariance $r(t)$ continue avec $r(0)=1$. Supposons aussi que les hypothèses (H_1) et (H_2) énoncées dans l'introduction sont satisfaites. Nous posons pour tout α compris entre 0 et 1 strictement

$$\forall t>1, \quad \varphi_\alpha(t)=(1-\alpha)\sqrt{2\log t},$$
$$\forall n \geqq 1, \quad T_n(\alpha)=\mathrm{Sup}\{|s-t|:]s, t[\subset]n, n+1[\cap \bar{A}_{\varphi_\alpha}(\omega)\}$$

où nous rappelons que $\overline{A_\varphi}(\omega)=\{u \geqq 0: |X(\omega, u)|>\varphi(u)\}$.

Avant d'énoncer le résultat principal de cet article, nous rappelons aussi un résultat relativement ancien dû à Pickands III (cf [2]) qui fait l'objet du

Lemme $\mathscr{P}$. *Sous les hypothèses précédentes, quel que soit α strictement positif, quel que soit $a>0$, φ_α appartient à la classe $\mathscr{L}_\infty(\{X(na)\}_{n \geq 1})$.*

Théorème 1. *Sous les hypothèses (H_1) et (H_2), il existe μ: $]0,1[\to]0, \infty[$ croissante avec $\mu(0)=0$, telle que pour tout α compris entre 0 et 1 strictement*

$$P\{\overline{\lim_{n \to \infty}} T_n(\alpha) \geqq \mu(\alpha)\}=1.$$

En particulier, $\forall 0<\alpha<1$, $\mu(X, \varphi_\alpha)>0$.

Démonstration. Fixons α et σ dans $]0,1[$ et posons

$$\varepsilon=\varepsilon(\alpha)=\frac{\alpha(3-\alpha)}{2(2-\alpha)}.$$

Soit $z(\varepsilon)$ défini au lemme 2 et notons τ_x la translation par x. Nous posons pour tout entier $n \geqq 1$

$$-\Omega_n'=\{\inf\{|X(s)|, s \in \tau_{n-1}(B(\sigma, z(\varepsilon))) \cap]n-1, n[\}<\varphi_\alpha(n)\}$$

$$-\Omega_n''=\left\{|X(\tau_{n-1}(\sigma))|>\frac{\varphi_\alpha(n)}{1-\varepsilon}\right\}.$$

Enfin μ est défini par

$$\forall 0<\alpha<1, \quad \mu(\alpha)=\inf(\lambda(B(\sigma, z(\varepsilon(\alpha))) \cap]0,1[), 0<\sigma<1\}.$$

La continuité de $r(t)$ assure $\mu(\alpha)>0$, pour $0<\alpha<1$.

Il résulte du lemme 2, pour tout n assez grand

$$P\{\Omega_n' \cap \Omega_n''\} \leqq C(\varepsilon)\, \Psi\left(\frac{1+\varepsilon}{1-\varepsilon}\, \varphi_\alpha(n)\right),$$

où $0<C(\varepsilon)<\infty$ ne dépend que de ε.

La majoration précédente ainsi que le choix de ε en fonction de α permettent de conclure à

$$\sum_{n=1}^{\infty} P(\Omega_n' \cap \Omega_n'')<\infty.$$

D'où, en vertu du lemme de Borel-Cantelli

$$\begin{aligned} 1 &= P\{\underline{\lim}_{n\to\infty}((\Omega'_n)^c \cup (\Omega''_n)^c)\}, \\ &\leqq P\{\overline{\lim}_{n\to\infty}(\Omega'_n)^c\} + 1 - P\{\overline{\lim}_{n\to\infty}\Omega''_n\}, \\ &= P\{\overline{\lim}(\Omega'_n)^c\}, \end{aligned}$$

car d'après le lemme $(\mathscr{P})$, $(1-\varepsilon)^{-1}\varphi_\alpha$ appartient à la classe $\mathscr{L}_\infty(\{X(\tau_n(\sigma))\}_{n\geq 1})$ de sorte que l'ensemble $\overline{\lim}_{n\to\infty}\Omega''_n$ a pour probabilité 1. Cette argumentation nous permet donc d'établir

$$P\{\overline{\lim}_{n\to\infty}(\Omega'_n)^c\} = 1,$$

et a fortiori

$$P\{\overline{\lim}_{n\to\infty} T_n(\alpha) \geqq \mu(\alpha)\} = 1.$$

§ 3. Exemples de processus gaussiens pour lesquels $\mu(X, \alpha)$ est grand

Dans ce paragraphe, nous étudions le comportement de μ en fonction de α et du processus gaussien stationnaire centré. Nous nous restreignons volontairement au cas où le processus vérifie les hypothèses suivantes

(H'_1) si F est le spectre de X, alors

$$\lambda_2 = \int_0^\infty \lambda^2 dF(\lambda) < \infty.$$

Cette condition est nécessaire et suffisante pour l'existence et la finitude de $r''(0)$, si $r(u) = \int_0^\infty \cos\lambda u\, dF(\lambda)$. On a alors

- $r''(0) = -\lambda_2$,
- pour tout $u \geqq 0$, $d^2(u) = 2(1-r(u)) \leqq \lambda_2 u^2$.

(H_2) $$\lim_{u\to\infty} r(u) = 0.$$

Ces processus ont des propriétés très remarquables qui sont étudiées avec soin dans l'important ouvrage [1] de Cramer et Leadbetter. Nous nous bornerons, pour notre part, à dire que le choix de ce type de processus est motivé par le fait que leurs lois possèdent de fortes dépendances locales. Cette particularité en fait de bons candidats pour notre étude.

Théorème 2. *Sous les hypothèses précédentes, il existe une constante $k_0 > 0$, absolue, (indépendante de X et de α), telle que*

$$\forall 0 < \alpha < 1, \quad \mu(X, \alpha) \geqq \sqrt{\frac{1}{-r''(0)}} \cdot g(k_0 \alpha),$$

où

$$g(t)=t|\log t|^{-\frac{1}{2}}, \quad t>0.$$

L'énoncé précédent montre que $\mu(X,\alpha)$ est inversement proportionnel à $\sqrt{-r''(0)}$, ce qui corrobore l'intuition.

Démonstration. Soit $T>0$ tel que $z=T\sqrt{-r''(0)}<\sqrt{2}$, et posons pour tout $u\geqq 0$

$$f(u)=u\sqrt{-r''(0)}.$$

Alors

$$\begin{aligned}
&-\forall u\geqq 0, \quad d(u)\leqq f(u),\\
(1)\qquad &-\sup_{0\leqq s,t\leqq T} d(|s-t|)\leqq z<\sqrt{2}\\
&-\forall a>0, \quad M_d([0,T],a)\leqq M_f([0,T],a)=\left[\frac{T\sqrt{-r''(0)}}{a}\right]=\left[\frac{z}{a}\right]\leqq\frac{\sqrt{2}}{a}.
\end{aligned}$$

Posons $\rho_0=\bar{e}^4$ et $H(T,z)=\dfrac{2}{\rho_0^2(1-\rho_0^2)}\displaystyle\int_0^{\rho_0 z}\left[u\vee\sqrt{\log\frac{M_d([0,T],u)}{u}}\right]du.$

On déduit de (1)

$$\begin{aligned}
H(T,z)&\leqq\frac{2}{\rho_0^2(1-\rho_0^2)}\int_0^{\rho_0 z}\left[u\vee\sqrt{\log\frac{\sqrt{2}}{u^2}}\right]du,\\
&\leqq\frac{2}{\rho_0^2(1-\rho_0^2)}\int_0^{\rho_0 z}\sqrt{\log\frac{\sqrt{2}}{u^2}}\,du,\\
&\leqq C_1 z\sqrt{\log\frac{1}{z}}, \quad \text{où } C_1>0 \text{ est une constante absolue.}
\end{aligned}$$

Ainsi si $C_2=C_1\sqrt{\frac{3}{2}}$, $g(t)=t\left(\log\dfrac{1}{t}\right)^{-\frac{1}{2}}$, $0<t\leqq 1$, et $z(\varepsilon)=g\left(\dfrac{\varepsilon}{C_2}\right)$; alors pour tout $0<z\leqq z(\varepsilon)$, nous avons

$$(2)\qquad H(T,z)\leqq\varepsilon.$$

Soit $0<\alpha<1$; nous posons maintenant

$$\mu_0(\alpha)=\inf\left\{\lambda(B_d(\sigma,z(\varepsilon))\cap[0,T]),\ 0<\sigma<T,\ \frac{2}{2-\alpha}<\varepsilon<\alpha\right\}$$

où λ est la mesure de Lebesgue sur $\mathbb{R}$.

Nous allons minorer $\mu_0(\alpha)$. Puisque de toute évidence

$$B_d(\sigma,z(\varepsilon))\supset B_f(\sigma,z(\varepsilon))=\left]\sigma-\frac{z(\varepsilon)}{\sqrt{-r''(0)}},\sigma+\frac{z(\varepsilon)}{\sqrt{-r''(0)}}\right[,$$

on en déduit

$$\begin{aligned}
\mu_0(\alpha)&\geqq\inf\left\{\frac{z(\varepsilon)}{\sqrt{-r''(0)}},\ \frac{\alpha}{2-\alpha}<\varepsilon<\alpha\right\}\\
&\geqq\frac{z(\alpha/2)}{\sqrt{-r''(0)}}.
\end{aligned}$$

Notons enfin pour tout $0<\alpha<1$, $\mu(\alpha)=(-r''(0))^{-\frac{1}{2}} g\left(\frac{\alpha}{2C_2}\right)$, et posons pour tout $n\geqq 1$, $0<\sigma<T$ et $\frac{\alpha}{2-\alpha}<\varepsilon<\alpha$,

$$\Omega'_n=\{\inf\{|X(s)|,\ s\in\tau_{(n-1)T}(B(\sigma,z(\varepsilon))\cap]0,T[)<\varphi_\alpha(n)\},$$

$$\Omega''_n=\left\{X(\tau_{(n-1)T}(\sigma))>\frac{\varphi_\alpha(n)}{1-\varepsilon}\right\}.$$

Il suffit à présent de reprendre la démonstration du théorème 1 en l'adaptant à notre situation, pour conclure au résultat.

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Sur les D-modules asymptotiques des processus strictement stationnaires ergodiques

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§1. Introduction

Dans cet article nous étudions quelques propriétés asymptotiques des processus strictement stationnaires $X(t)$, t réel. Cette étude est essentiellement motivée par le travail récent de D. Geman ([3]) ainsi que par le nôtre ([4, 5]). L'idée de base de ([3]) consiste en la recherche de modules $\varphi(t)$, $t>0$, contrôlant le comportement asymptotique du processus dans le sens que l'on ait, avec probabilité 1,

$$\frac{|X(t)|}{\varphi(t)} \leqq 1,$$

non pas pour tout t suffisamment grand, mais pour tout t dans un ensemble «plein» à l'infini. Posons à cet effet pour tout ensemble $A \subset (0, \infty)$, Lebesgue mesurable, et tout $t>0$,

$$h_A(t) = \frac{m(A \cap (0,t))}{t},$$

où m est la mesure de Lebesgue. La densité de A à l'infini est la limite quand elle existe $D_\infty(A) = \lim_{t \uparrow \infty} h_A(t)$.

Cette définition a été introduite dans [3]. C'est une des notions possibles de densité à l'infini. Rappelons que la densité de A en un point $0 \leqq \xi < \infty$ est la limite quand elle existe

$$D_\xi(A) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} m(A \cap (\xi - \varepsilon, \xi + \varepsilon)).$$

Le théorème suivant teste la nullité de $D_\infty(A)$, (resp. $D_0(A)$).

Théorème ([3], p. 8). *Pour tout $A \subset (0, \infty)$, Lebesgue mesurable, $D_0(A)=0$, (resp. $D_\infty(A)=0$), si et seulement si, il existe $\Psi(x)$, $0 \leqq x \leqq 1$, continue, convexe,*

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croissante, avec $\Psi(0)=0$, *telle que*

$$\int_0^1 I_A(t)\,\Psi_0 h_A(t)\frac{dt}{t}<\infty, \qquad \left(\text{resp.}\int_1^\infty I_A(t)\,\Psi_0 h_A(t)\frac{dt}{t}<\infty\right).$$

On dit que 0 (resp. (∞)) est un point de dispersion de A lorsque $D_0(A)=0$ (resp. $D_\infty(A)=0$).

Soit $F(t)$, $0\leqq t\leqq\infty$, mesurable; on définit aussi

$$(D)\limsup_{t\to\infty(0)} F(t)=\inf\{l>0: D_{\infty(0)}\{F>l\}=0\}.$$

Considérons un processus gaussien centré, mesurable, $X(t)$, $t\geqq 0$ tel que $E\left\{\frac{X(s)}{\sigma(s)}\cdot\frac{X(t)}{\sigma(t)}\right\}\to 0$ quand $|s-t|\to\infty$, où pour tout $u>0$, $\sigma^2(u)=E\{X^2(u)\}>0$. Posons alors pour tout $t\geqq e$, $L(t)=\sqrt{2\operatorname{Log}\log t}$. A l'aide du théorème précédent, D. Geman a montré ([3]), th. 9.1),

$$(D)\lim_{t\to\infty}\frac{|X(t)|}{\sigma(t)\,L(t)}=0 \quad \text{p.s.} \tag{1}$$

Par ailleurs, sans hypothèse sur la covariance du processus, on a trivialement

$$(D)\limsup_{t\to\infty}\frac{|X(t)|}{\sigma(t)\sqrt{2\operatorname{Log} t}}\leqq 1. \tag{2}$$

Enfin, soit $X(t)$, $t\geqq 0$ un processus gaussien stationnaire centré, de covariance $r(t)=E\{X(u)\cdot X(u+t)\}$ continue avec $r(0)=1$, telle que

$$r(u)=O(1/\operatorname{Log} u)\,u\to\infty.$$

Soit aussi $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ croissante telle que $I(\varphi)=\int^{+\infty}\Psi_0\varphi(u)\,du=\infty$, (où pour tout $u\geqq 0$, $\Psi(u)=P\{(\mathcal{N}(0,1)>u\}$), et $\varphi(t)\overset{\cup}{\cap}\sqrt{\log t}$ quand t tend vers l'infini. Nous avons montré dans [4], (théorème 7.1),

$$\limsup_{T\to\infty}\frac{m\{0\leqq t\leqq T: X(t)>\varphi(t)\}}{\int_0^T\Psi_0\varphi(u)\,du}\geqq 1 \quad \text{p.s.} \tag{3}$$

Nous nous proposons dans ce qui suit de renforcer très nettement (1) et (2) lorsque $X(t)$ est un processus strictement stationnaire ergodique, mesurable et séparable pour la topologie usuelle. Nous rappelons qu'un processus strictement stationnaire est ergodique pour la transformation $X(t)\xrightarrow{T}X(t+1)$ lorsque $(\mathcal{S}(X),T)$ est un système dynamique ergodique ($\mathcal{S}(X)$ est l'espace de probabilité canonique du processus X). Nous utiliserons essentiellement la notion de système induit et notre ouvrage de référence est [1].

§2. Lemmes préliminaires

Lemme 2.1. *Soit $(E, \mathscr{B}, \mu, T)$ un système dynamique mesuré. On suppose que la transformation T est inversible et ergodique pour μ. Notons alors pour tout A élément de $\mathscr{B}$ tel que $\mu(A)>0$,*

$$\forall f \in A, \quad k^1(f)=\inf\{k>0: T^k f \in A\},$$
$$\forall j>1, \quad k^j(f)=\inf\{k>k^{j-1}(f): T^k f \in A\}.$$

Alors si $\mu_A(\cdot)=\mu(\cdot|A)$, on a

$$\mu_A\left\{\lim_{j\to\infty}\frac{k^j}{j}=\frac{1}{\mu(A)}\right\}=1.$$

Démonstration. Posons $\mathscr{B}_A=\mathscr{B}\cap A$, et pour tout f dans A, $T_A(f)=T^{k^1(f)}(f)$. On sait, (cf. [1] p. 130) que le système induit $(A, \mathscr{B}_A, T_A, \mu_A)$ est ergodique. Considérons l'application mesurable $F=k^1:(A,\mathscr{B}_A)\to\mathbb{N}$.

On constate que F est intégrable. En effet,

$$E_A(F)=\int_A F\,d\mu_A=\sum_{l=1}^{\infty}\mu_A(F\geqq l),$$

mais pour tout $l\geqq 1$,

$$\mu_A(F\geqq l)=\frac{\mu(A\cap(T^{-1}A)^c\cap\ldots\cap(T^{-(l-1)}A)^c)}{\mu(A)},$$
$$=\frac{\mu(T^{(l-1)}A\cap(T^{(l-2)}A)^c\cap\ldots\cap A^c)}{\mu(A)},$$

de sorte que

$$E_A(F)=\frac{\mu\left(\bigcup_{k=0}^{\infty}T^kA\right)}{\mu(A)}=\frac{1}{\mu(A)},$$

car $\bigcup_{k=0}^{\infty}T^kA$ est invariant par T, de mesure strictement positive, donc égale à 1 puisque T est ergodique pour μ. Le théorème de Birkhoff-Khintchine, appliqué au système dynamique ergodique (A, T_A, μ_A), permet donc d'aboutir au résultat cherché puisque F appartient à $L^1(A,\mu_A)$.

Lemme 2.2. *Soit $\{X(t), t\in\mathbb{R}\}$ un processus stochastique, strictement stationnaire et ergodique pour la transformation $X(t)\xrightarrow{T}X(t+1)$. Si de plus X est mesurable et séparable pour la topologie usuelle, alors pour tout élément A de la tribu produit de $\mathbb{R}^{\mathbb{R}}$, notons*

$$T_0(\omega)=I_A(X(\omega)),$$
$$\forall n\geqq 1,\ T_n(\omega)=\inf\{k>T_{n-1}(\omega): T^k(X(\omega))\in A\},$$

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[1] $f(t)\underset{\cap}{\cup}g(t)$ en 0 (resp. ∞) lorsque $0<\liminf_{t\to 0(\infty)}\left|\frac{f(t)}{g(t)}\right|\leqq\limsup_{t\to 0(\infty)}\left|\frac{f(t)}{g(t)}\right|<\infty$

Si $P\{T_0>0\}>0$, *nous avons aussi,*

$$P\left\{\lim_{j\to\infty}\frac{T_j}{j}=\frac{1}{P\{T_0>0\}}\middle|T_0>0\right\}=1.$$

Démonstration. C'est une conséquence immédiate du lemme précédent.

Lemme 2.3. *Soit* $\{X(t), t\in T\}$ *un processus gaussien centré séparable pour l'écart* $d^2(s,t)=E\{(X(s)-X(t))^2\}$. *On suppose aussi*

(h_1) $\forall t\in T, E\{X^2(t)\}=1$,

(h_2) *si pour tout* $u>0$, $M_d(T,u)$ *est le cardinal maximal des suites* $S\subset T$ *telles que,* $\forall s,t\in S$, $s\neq t\Rightarrow d(s,t)>u$, *l'intégrale* $\int_{+0}\sqrt{\operatorname{Log} M_d(T,u)}\,du$ *converge.*

Alors pour tout ε *compris entre* 0 *et* 1 *strictement, on peut trouver deux nombres* $0<B(\varepsilon)<\infty$ *et* $0<z(\varepsilon)<\infty$ *tels que quels que soient* $\lambda>\sqrt{2}$, $0<z\leqq z(\varepsilon)$ *et* σ *élément de* T,

$$P\{\inf\{|X(s)|, s\in B_d(\sigma,z)\}<\lambda(1-\varepsilon), |X(\sigma)|>\lambda\}\leqq B(\varepsilon)\,\Psi(\lambda(1+\varepsilon)),$$

où $\forall t\geqq 0\ \Psi(t)=P\{n(0,1)>t\}$.

Ce lemme est démontré dans [5].

Corollaire 2.4. *Sous les hypothèses du lemme précédent, il existe* $a>0$, *tel que, quels que soient* $\lambda>0$ *et* $\sigma\in T$,

$$0<P\{\inf\{|X(s)|s\in B(\sigma,a)\}<\lambda\}<1.$$

Démonstration. Choisissons ε compris entre 0 et 1 strictement, et soit $z(\varepsilon)=a$. Le lemme précédent montre

$$\begin{aligned}P\{\inf\{|X(s)|, s\in B(\sigma,a)\}<\lambda(1-\varepsilon)\}&\leqq B(\varepsilon)\,\Psi(\lambda(1+\varepsilon))+P\{|X(\sigma)|<\lambda\}\\&=B(\varepsilon)\,\Psi(\lambda(1+\varepsilon))+1-2\Psi(\lambda)<1,\end{aligned}$$

pour tout λ suffisamment grand.

On en déduit pour tout $\lambda'>0$,

$$0<P\{\inf\{|X(s)|, s\in B(\sigma,a)\}<\lambda'\}<1.$$

Lemme 2.5. *Soit* $X(t)$, t *réel, un processus strictement stationnaire, ergodique pour la transformation* $X(t)\xrightarrow{T}X(t+1)$, *mesurable et séparable pour la topologie usuelle. Alors pour tout* $x>0$,

$$P\left\{\liminf_{U\to\infty}\frac{m\{0\leqq u\leqq U: |X(\omega,u)|\leqq x\}}{U}\geqq(1-\sqrt{G(x)})^2\right\}=1,$$

où $G(x)=P\{|X(0)|>x\}$.

Démonstration. Il suffit de traiter le cas $G(x)<1$.

Posons pour tout $x>0$ et $0<\gamma<1$,

- $\alpha(\gamma,x)=\{\omega: m\{0\leqq t\leqq 1: |X(\omega,t)|\leqq x\}>\gamma\}$
- $\gamma_x=1-\sqrt{G(x)}$.

Alors, par suite de la mesurabilité de X, pour tout $0<\gamma<\gamma_x$,

$$\begin{aligned} P\{\alpha(\gamma,x)\} &= P\{m\{0\leqq t\leqq 1: |X(\omega,t)|>x\}\leqq 1-\gamma\} \\ &= 1-P\{m\{0\leqq t\leqq 1: |X(\omega,t)|>x\}>1-\gamma\} \\ &\geqq 1-\frac{E[m\{0\leqq t\leqq 1: |X(\omega,t)|>x\}]}{1-\gamma} \qquad (1) \\ &= 1-\frac{G(x)}{1-\gamma}>0 \end{aligned}$$

Fixons maintenant $x>0$ et $0<\gamma<\gamma_x$ de façon à ce que $P(\alpha(\gamma,x)\}>0$. On constate à partir de (1)

$$\lim_{x\to\infty} P\{\alpha(\gamma,x)\}=1. \qquad (2)$$

Définissions la suite des temps de retour du processus dans $\alpha(\gamma,x)$,

$$\begin{aligned} &-R_0(\omega)=I_{\alpha(\gamma,x)}(\omega) \\ &-\forall j>0, R_j(\omega)=\inf\{k>R_{j-1}(\omega): T^k(X(\omega))\in\alpha(\gamma,x)\}, \end{aligned} \qquad (3)$$

où $T^k(X(\omega))=\underbrace{T_0\,T_0\cdots_0 T}_{k\text{ fois}}(X(\omega))=X(\omega,\cdot+k)$.

Le lemme 2.2, appliqué à notre situation, établit

$$P\left\{\lim_{j\to\infty}\frac{R_j}{j}=\frac{1}{P\{\alpha(\gamma,x)\}}\,\middle|\,R_0>0\right\}=1. \qquad (4)$$

De plus, pour tout n entier et tout ω dans $\alpha(\gamma,x)$, nous avons

$$\begin{aligned} &m\{0\leqq t\leqq 1: |X(\omega,R_n(\omega)+t)|\leqq x\} \\ &=m\{R_n(\omega)\leqq t\leqq R_n(\omega)+1: |X(\omega,t)|\leqq x\}\geqq\gamma. \end{aligned} \qquad (5)$$

Soit maintenant ω dans $\alpha(\gamma,x)\cap\left\{\lim_{j\to\infty}\frac{R_j}{j}=\frac{1}{P\{\alpha(\gamma,x)\}}\right\}$. Pour tout $\varepsilon>0$, il existe $j(\varepsilon,\omega)<\infty$, tel que

$$\forall j>j(\varepsilon,\omega), j\left\{\frac{1}{P\{\alpha(\gamma,x)\}}-\varepsilon\right\}\leqq R_j<j\left\{\frac{1}{P(\alpha(\gamma,x)\}}+\varepsilon\right\}. \qquad (6)$$

Soit alors $t>R_{j(\varepsilon,\omega)}$ et j l'entier tel que

$$R_j+1\leqq t<R_{j+1}+1. \qquad (7)$$

On a par inclusion

$$\frac{m\{0\leqq u\leqq t:|X(\omega,u)|\leqq x\}}{t}\geqq\frac{m\{0\leqq u\leqq R_j+1:|X(\omega,u)|\leqq x\}}{R_{j+1}+1}. \tag{8}$$

$$\geqq\frac{\gamma j}{(j+1)\left(\dfrac{1}{P\{\alpha(\gamma,x)\}}+\varepsilon\right)+1}$$

Cette dernière inégalité montre

$$\liminf_{t\to\infty}\frac{m\{0\leqq u\leqq t:|X(\omega,u)|\leqq x\}}{t}$$

$$\geqq\frac{\gamma}{\dfrac{1}{P\{\alpha(\gamma,x)\}}+\varepsilon}. \tag{9}$$

Mais comme $\varepsilon>0$ est arbitraire, on obtient donc pour tout ω dans $\alpha(\gamma,x)\cap\left\{\lim_{j\to\infty}\dfrac{R_j}{j}=\dfrac{1}{P\{\alpha(\gamma,x)\}}\right\}$, ensemble de probabilité strictement positive,

$$L(\omega)=\liminf_{t\to\infty}\frac{m\{0\leqq u\leqq t:|X(\omega,u)|\leqq x\}}{t}\geqq\gamma P\{\alpha(\gamma,x)\}$$

$$\geqq\gamma\left(1-\frac{G(x)}{1-\gamma}\right). \tag{10}$$

Mais puisque X est ergodique, la v.a.r. L a une loi dégénérée. On a donc pour tout $x>0$ *et tout* $0<\gamma<\gamma_x$,

$$\text{p.s. } L(\omega)\geqq\gamma\left(1-\frac{G(x)}{1-\gamma}\right), \tag{11}$$

d'où, en faisant tendre γ vers γ_x,

$$P\left\{\liminf_{t\to\infty}\frac{m\{0\leqq u\leqq t:|X(\omega,u)|\leqq x\}}{t}\geqq(1-\sqrt{G(x)})^2\right\}=1, \tag{12}$$

ce qui établit le lemme 2.5.

Le lemme suivant se démontre exactement de la même façon que le lemme précédent; nous en omettons donc la démonstration.

Lemme 2.6. *Sous les hypothèses du lemme* 2.5 *pour tout* $x>0$, *nous avons*

$$P\left\{\liminf_{U\to\infty}\frac{m\{0\leqq u\leqq U:|X(\omega,u)|\geqq x\}}{U}\geqq(1-\sqrt{1-G(x)})^2\right\}=1,$$

où $G(x)=P\{|X(0)|>x\}$.

La conjugaison des deux résultats précédents a pour conséquence le

Corollaire 2.7. *Sous les hypothèses du lemme* 2.5, *l'oscillation à l'infini de la variable aléatoire*

$$L_x(\omega, U)=\frac{m\{0\leqq t\leqq U: |X(\omega,t)|\leqq x\}}{U},$$

est presque sûrement majorée par $2(\sqrt{1-G(x)}+\sqrt{G(x)}-1)$. *En d'autres termes,*

p.s. $$W_\infty(L_x(\omega))=\lim_{U\to\infty}\sup_{t,t'>U}|L_x(\omega,t)-L_x(\omega,t')|$$
$$\leqq 2(\sqrt{1-G(x)}+\sqrt{G(x)}-1).$$

Ce corollaire montre donc lorsque $P\{X(0)=0\}=0$,

p.s. $$\lim_{x\to 0} W_\infty(L_x(\omega))=\lim_{x\to\infty} W_\infty(L_x(\omega))=0.$$

§3 Applications

Les théorèmes 1 et 2 qui suivent concernent les *D*-modules asymptotiques des processus strictement stationnaires ergodiques. Ils montrent que la notion de *D*-module est beaucoup trop vague pour décrire le comportement asymptotique de ces processus.

Théorème 3.1. *Soit* $X(t), t$ *réel un processus strictement stationnaire, mesurable, séparable pour la topologie usuelle et ergodique pour la transformation* $X(t)\xrightarrow{T}X(t+1)$. *Alors pour toute application croissante* $g:\mathbb{R}_+\to\mathbb{R}_+$, *telle que* $\lim_{t\to\infty} g(t)=\infty$, *on a*

$$(D)\ \lim_{t\to\infty}\frac{|X(t)|}{g(t)}=0 \quad \text{p.s.}$$

Démonstration. Il suffit d'établir pour toute application croissante g telle que $\lim_{t\to\infty} g(t)=\infty$,

$$D_\infty(\{t\geqq 0:|X(t)|>g(t)\})=0 \quad \text{p.s.}, \tag{1}$$

ou

$$\lim_{t\to\infty}\frac{m\{0\leqq u\leqq t:|X(u)|\leqq g(u)\}}{t}=1 \quad \text{p.s.} \tag{1'}$$

Soient $g(t)$ fixé et $x>0$. Notons $T=T(g,x)$ l'entier tel que

$$\forall t>T,\ g(t)>x. \tag{2}$$

On a par inclusion, pour tout $t>0$,

$$\begin{aligned}
&\frac{m\{0\leqq u\leqq T+t:|X(\omega,u)|\leqq g(u)\}}{t+T},\\
&\geqq\frac{m\{T\leqq u\leqq T+t:|X(\omega,u)|\leqq g(u)\}}{t+T},\\
&\geqq\frac{m\{T\leqq u\leqq T+t:|X(\omega,u)|\leqq x\}}{t+T},\\
&=\frac{m\{0\leqq u\leqq T+t:|X(\omega,u)|\leqq x\}}{t+T}-\frac{m\{0\leqq u\leqq T:|X(\omega,u)|\leqq x\}}{t+T}.
\end{aligned} \tag{3}$$

Nous déduisons du lemme 2.5 en faisant tendre t vers l'infini

$$\text{p.s.} \quad \liminf_{U=T+t\to\infty} \frac{m\{0\leqq u\leqq U: |X(\omega,u)|\leqq g(u)\}}{U} \geqq (1-\sqrt{G(x)})^2, \tag{4}$$

expression dans laquelle x est devenu arbitraire. On en déduit, en faisant tendre x vers l'infini

$$\text{p.s.} \quad \lim_{U\to\infty} \frac{m\{0\leqq u\leqq U: |X(\omega,u)|\leqq g(u)\}}{U} = 1. \tag{5}$$

Ce qui montre pour toute application croissante $g(t)$ telle que $\lim_{t\to\infty} g(t)=\infty$,

$$\text{p.s.} \quad D_\infty(\{t\geqq 0: |X(t)|>g(t)\})=0. \tag{6}$$

Ce dernier point exprime

$$(D)\lim_{t\to\infty} \frac{|X(t)|}{g(t)}=0 \quad \text{p.s.}, \tag{7}$$

ce qui est le résultat cherché.

Théorème 3.2. *Sous les hypothèses du théorème* 3.1, *si de plus* $P\{X(0)=0\}=0$, *alors pour toute application décroissante* $h: \mathbb{R}_+ \to R_+$, *telle que* $\lim_{t\to\infty} h(t)=0$, *on a*

$$(D)\lim_{t\to\infty} \frac{|X(t)|}{h(t)}=\infty \quad \text{p.s.}$$

La démonstration de ce théorème recopie en l'adaptant celle du théorème 3.1. aussi nous l'omettons.

§4. Variante

Dans ce paragraphe, nous supposons que le processus étudié est partiellement majoré au sens suivant

($\mathscr{C}$) il existe $a>0$ tel que notant $\|X\|_a=\sup\{|X(t)|, 0\leqq t\leqq a\}$,

on a $P\{\|X\|_a<\infty\}>0$.

Notons alors $\mathscr{S}(a)$ l'ensemble des familles dénombrables d'intervalles $I=\{I_n, n\geqq 1\}$ deux à deux disjoints de longueur commune a. Pour tout $0\leqq\rho\leqq 1$, nous posons aussi

$$\mathscr{S}(a,\rho)=\{I\in\mathscr{S}(a): D_\infty(I)=\rho\}.$$

Le lemme 2.2 permet directement de majorer ou de minorer asymptotiquement les trajectoires de X non plus pour tout t dans un ensemble de densité de Lebesgue égale à 1, mais pour tout t parcourant une suite (aléatoire) I dans $\mathscr{S}(a,\rho)$, $\rho>0$.

Dans la proposition suivante nous avons noté

$$\forall x>0, \forall a>0, A(a,x)=\{\|X\|_a \leqq x\}.$$

Proposition 4.1 (Majoration). *Supposons que X vérifie les hypothèses du théorème* 3.1. *ainsi que la condition* ($\mathscr{C}$). *Alors pour tout x tel que $P\{A(a,x)\}>0$, nous avons*

$$P\{\omega: \exists I \in \mathscr{S}(a, P\{A(a,x)\}): \forall t \in I, |X(\omega,t)| \leqq x\}=1.$$

Remarque. Ceci montre que X est presque sûrement majoré par x sur une suite d'intervalles dont la répartition est justement précisée par $P(\|X\|_a \leqq x)$.

Démonstration. Soit T_a la transformation $X(t) \to X(t+a)$, où $a>0$ est déterminé par ($\mathscr{C}$). Notons $\{R_n, n \geqq 0\}$ la suite des temps de retour dans $A(a,x)$, avec $R_0 = I_{A(a,x)}$ et $P\{A(a,x)\}>0$.

Pour tout ω dans $A(a,x)$, à un ensemble négligeable près, nous avons à partir du lemme 2.2

$$\lim_{n\to\infty} \frac{R_n}{n} = \frac{1}{P\{A(a,x)\}}. \tag{1}$$

Ainsi pour t parcourant $I(\omega)=\{[aR_n(\omega),\ a(R_n(\omega)+1)],\ n \geqq 0\}$, nous obtenons $|X(\omega,t)| \leqq x$. Il suit donc, puisque $D_\infty(I(\omega)) = \dfrac{1}{P\{A(a,x)\}}$,

$$P\{\omega: \exists I \in \mathscr{S}(a, P\{A(a,x)\}): \forall t \in I, |X(\omega,t)| \leqq x\} \geqq P\{A(a,x)\}>0. \tag{2}$$

Mais, puisque l'événement décrit en (2) est invariant par T_a, l'ergodicité du processus implique

$$P\{\omega: \exists I \in \mathscr{S}(a, P\{A(a,x)\}): \forall t \in I, |X(\omega,t)| \leqq x\}=1, \tag{3}$$

ce qui termine la démonstration.

De la proposition précédente nous déduisons facilement le

Corollaire 4.2. *Si les hypothèses de la proposition* 4.2. *sont renforcées en* ($\mathscr{C}'$): *il existe $a>0$ tel que $P\{\|X\|_a < \infty\}=1$.*

Alors pour toute application croissante $g(t)$ telle que $\lim_{t\to\infty} g(t)=\infty$, *nous avons*

$$\forall 0<\rho<1, P\{\omega: \exists I \in \mathscr{S}(a,\rho): \forall t \in I, |X(\omega,t)| \leqq g(t)\}=1.$$

n.b. Dans le cas gaussien les conditions ($\mathscr{C}$) et ($\mathscr{C}'$) sont équivalentes. De plus a est arbitraire.

La proposition suivante se démontre de la même façon que la proposition 4.1. Nous en omettons donc la démonstration.

Proposition 4.3. (Minoration, Cas gaussien). *Soit $\{X(t), t \in \mathbb{R}\}$ un processus gaussien stationnaire centré, séparable de covariance $r(u)=\int_0^\infty \cos \lambda u \, dF(\lambda)$, continue*

telle que

a) *la fonction spectrale F est dépourvue de sauts,*

b) *il existe un voisinage V de 0 tel que l'intégrale* $\int_{+0} \sqrt{\operatorname{Log} M_d(V,u)}\,du$ *converge,* (*où* $d(t)=[2(1-r(t))]^{\frac{1}{2}}$).

Alors il existe $a>0$ *tel que si* $\beta(a,x)=\{\inf\{|X(s)|,\ 0\leqq s\leqq a\}\geqq x\}$, *on ait*

i) $\forall x>0, P\{\beta(a,x)\}>0$,

ii) $\forall x>0, P\{\omega: \exists I\in\mathscr{S}(a, P\{\beta(a,x)\}): \forall t\in I, |X(\omega,t)|\geqq x\}=1$.

n.b. La condition a) est la condition nécessaire et suffisante pour l'ergodicité de X. La condition b) caractérise la continuité des trajectoires de X, (cf [2], p. 89). Enfin i) se déduit du corollaire 2.4.

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Reçu le 22 Janvier 1980

Graphs and Questionnaires

by **CLAUDE FRANÇOIS PICARD,** *Centre National de la Recherche Scientifique, Paris, France.*

NORTH-HOLLAND MATHEMATICS STUDIES, Volume 32

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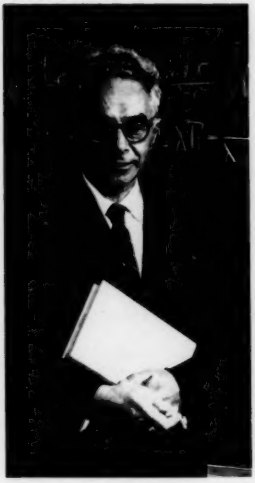

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Sanserif	A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z
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preferred form	instead of	preferred form	instead of
A^*, $\tilde{b}, \gamma', \mathfrak{v}, \mathbf{v}$	$\bar{A}, \hat{b}, \tilde{\gamma}, \vec{v}$	$f\colon A \to B$	$A \xrightarrow{f} B$
lim sup, lim inf	$\overline{\lim}$, $\underline{\lim}$		
inj lim, proj lim	$\varinjlim$, $\varprojlim$		
$\exp(-(x^2+y^2)/a^2)$	$e^{-\frac{x^2+y^2}{a^2}}$	$\dfrac{\cos(1/x)}{(a+b/x)^{1/2}}$	$\dfrac{\cos\frac{1}{x}}{\sqrt{a+\frac{b}{x}}}$
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